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Infinitesimal symmetries: a computational approach

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Preface

This monograph is concerned with computational aspects in the determination of infinitesimal symmetries and Lie-Bäcklund transformations of differential equations. Moreover some problems are calculated explicitly.

A brief introduction to some concepts in the theory of symmetries and Lie-Bäcklund transformations, relevant for this book, are given. The mathematical formalism is shortly reviewed. The jet bundle formulation is chosen, in which, by its algebraic nature, objects can be described very precisely. Consequently it is appropriate for implementation.

A number of procedures are discussed, which enable to carry through computations with the help of a computer. These computations are very extensive in practice.

The Lie algebras of infinitesimal symmetries of a number of differential equations in Mathematical Physics are established and some of their applications are discussed, i.e., Maxwell equations, nonlinear diffusion equation, nonlinear Schrödinger equation, nonlinear Dirac equations and self dual SU(2) Yang-Mills equations.

Lie-Bäcklund transformations of Burgers' equation, Classical Boussinesq equation and the Massive Thirring Model are determined. Furthermore non-local Lie-Bäcklund transformations of the last equation are derived.

I would like to thank Ruud Martini for the exemplary way in which he stimulated my research in the past four years. Without his interest and inspiration this would not have been accomplished.

I want to thank Peter Gragert, with whom I shared my enthusiasm in achieving result, for his valuable contribution to chapter 2.

It was gratifying to see that Manuela de Mol-Fernández discovered symmetry in symmetries. I would like to express appreciation for the way in which she took care for typing the manuscript.

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Contents

INTRODUCTION

In this tract we are concerned with the computational aspects of infinitesimal symmetries and Lie-Bäcklund transformations, which are amongst others useful expedients in the study of nonlinear differential equations.

In section 1 of chapter 1 we give a short survey of the notions of local jet bundles, differential equations, contact modules, exterior differential systems, symmetries and similarity solutions.

The notion of Lie-Bäcklund transformations is introduced in section 1.2.

A very short introduction to nonlocal symmetries by means of an example describing nonlocal symmetries of Burgers' equation is given in section 1.3.

In chapter 2 we give a description of the software which has been developed to calculate infinitesimal symmetries and Lie-Bäcklund transformations on a DEC system 20 using the symbolic language REDUCE.

In section 2.1 we indicate what kind of procedures are needed to carry through the computation. It is our intention that readers not familiar with LISP should be able to understand section 2.2 where a general description of the software is given. More details of the construction are given in section 2.3 where we make references to the source code which is at the end of this tract. We think that this part is suited for readers having some knowledge of LISP.

Finally, in section 2.4 we give a copy of a complete computer session of the calculation of the Lie algebra of infinitesimal symmetries of Burgers' equation.

In chapter 3 we give some of the results which we obtained in the computation of infinitesimal symmetries of partial differential equations of Mathematical Physics, using the software described in chapter 2 i.e., vacuum Maxwell equations, nonlinear diffusion equation, nonlinear Schrödinger equation. In section 3.4 we compute the Lie algebra of infinitesimal symmetries of 4 types of (non)linear Dirac equations and construct conserved currents associated to new infinitesimal symmetries.

In section 2.5 we compute Lie algebra of infinitesimal symmetries of

self dual SU(2) Yang-Mills equations and obtain the Belavin-Polyakov-Schwartz-Tyupkin instanton solution as a similarity solution. The Lie algebra of infinitesimal symmetries of the static self dual SU(2) gauge field is given and the Prasad-Sommerfield monopole solution is obtained as a similarity solution too.
To demonstrate the technique of computing Lie-Bäcklund transformations we study Burgers' equation in section 4.1.
In section 4.2 we compute Lie-Bäcklund transformations of the Classical Boussinesq equation yielding an (x,t)-dependent Lie-Bäcklund transformation whose action on (x,t)-independent Lie-Bäcklund transformations is similar to the action of a recursion operator recently discovered for this equation. Second and third order Lie-Bäcklund transformations of the Massive Thirring Model are obtained in section 4.3.
The search for a generating operator for the Massive Thirring Model was the motivation for section 4.4.
After a brief introduction to nonlocal Lie-Bäcklund transformations we compute 2 nonlocal Lie-Bäcklund transformations which act on the (x,t)-independent Lie-Bäcklund transformations as creating and annihilating operators.

Chapter 1

INTRODUCTION TO INFINITESIMAL SYMMETRIES AND LIE-BACKLUND TRANSFORMATIONS

1.0 Introduction

In this chapter some of the general notions of the theory of infinitesimal symmetries and Lie-Bäcklund transformations are briefly reviewed.
More comprehensive descriptions are given in [1], [9], [29], [30], [34], [39].
In [1], [29], [39], considerations are based on vector fields, whereas in [9] [30], [34] most considerations are based on differential forms.
We assume that the reader is familiar with elementary notions of differential geometry, such as differential forms, exterior derivative, vector fields, Lie derivative, contraction of a differential form and a vector field.
The complete discussion will be of local nature and we shall study only regular cases.

In section 1 a short introduction to the local jet bundle formulation is given according to [30], [34].
(Partial) differential equations and exterior differential systems are defined. The notion of a symmetry of an exterior differential system is introduced and the infinitesimal criterion is given.
A theorem concerning infinitesimal symmetries due to Bäcklund is adopted from [1], Noether's theorem [27] is given for later use in chapter 4; similarity solutions are introduced. An elegant theorem due to Kumei & Bluman [23] is formulated and applied to Thomas' equation [38].
In section 2 Lie-Bäcklund transformations are defined as vector fields on infinite jet bundles. The equivalence between Lie-Bäcklund transformations is introduced and the criterion for a vector field to be a Lie-Bäcklund transformation is given. Finally in section 3 we indicate how the notion of infinitesimal symmetries can be generalized by introduction of nonlocal variables.

1.1 Local jet bundles, differential equations, exterior differential systems, symmetries

Some of the general notions of the local jet bundle formulation are given. A more extensive introduction is given in [30], [34].
We shall adopt the notation from [34].
Partial differential equations are described according to [34] and exterior differential systems on a jet bundle are introduced. The prolongation of a differential equation and of an exterior differential system is defined.
Symmetries of exterior differential systems are defined and the criterion for a vector field to generate a local 1-parameter group of symmetries is given.
A theorem concerning the structure of the components of infinitesimal symmetries due to Bäcklund [1] is given.
For later use in chapters 3 & 4 Noether's theorem is given and the notion of similarity solutions [5], [12] is introduced.

Let M,N be C^∞ manifols and let $C^\infty(M,N)$ denote the collection of local C^∞ maps $f : U \to N$, U open in M.
In applications, chapter 3 & 4, M is the space of independent variables and N is the space of dependent variables.
Two maps $f,g \in C^\infty(M,N)$ are said to agree to order k at $x \in M$ if $f(x) = g(x)$ and if there are local charts around $x \in M$, and around $f(x) = g(x) \in N$ in which all derivatives at x, up to and including order k are the same.
The equivalence class of maps which agree with f to order k at x is called the k-jet of f at x, denoted $j^k_x f$.
If $x^a(a=1,\dots,\dim M)$ are local coordinates around $x \in M$ and $z^\mu(\mu=1,\dots,\dim N)$ are local coordinates around $f(x) \in N$, then $j^k_x f$ is determined by the quantities

$$x^a, z^\mu = f^\mu(x),\ z^\mu_a = \partial_a f^\mu(x), \dots,\ z^\mu_{a_1\dots a_k} = \partial_{a_1\dots a_k} f^\mu(x), \qquad (1.1.1)$$

where $f^\mu(x)$ is the coordinate presentation of f.

In (1.1.1) and further $\partial_a, \dots, \partial_{a_1\dots a_k}$, denote partial derivatives

$$\partial_a f^\mu(x) = \frac{\partial}{\partial x^a} f^\mu(x),\ \partial_{a_1\dots a_k} f^\mu(x) = \frac{\partial^k}{\partial x^{a_1}\dots\partial x^{a_k}} f^\mu(x). \qquad (1.1.1a)$$

Latin indices $a, a_1, \ldots$ range and sum over $1, \ldots, \dim M$, while Greek indices range and sum over $1, \ldots, \dim N$.

Conversely, any collection of numbers

$$x^a,\ z^\mu,\ z^\mu_a, \ldots, z^\mu_{a_1 \ldots a_k}$$

where $z^\mu_{a_1 \ldots a_\ell}$ are symmetric in their subscripts, determines a unique equivalence class.

From this we obtain the following definition of the <u>k-jet bundle</u>.

The k-jet bundle of M and N, denoted $J^k(M,N)$ is the set of all k-jets $j^k_x f$ with k fixed, $x \in M$, $f \in C^\infty(M,N)$, provided with a natural differentiable structure.

The map

$$\alpha : J^k(M,N) \to M \tag{1.1.2}$$

defined by

$$j^k_x f \mapsto x \tag{1.1.2a}$$

is called the source map, and x is called the source of $j^k_x f$.

The map

$$\beta : J^k(M,N) \to N \tag{1.1.3}$$

defined by

$$j^k_x f \mapsto f(x) \tag{1.1.3a}$$

is called the target map, and f(x) is called the target of $j^k_x f$.

In [34] it is demonstrated that

$$x^a, z^\mu, z^\mu_a, \ldots, z^\mu_{a_1 \ldots a_k} \tag{1.1.4}$$

may be chosen as local coordinates around $\xi \in J^k(M,N)$ where x^a are local coordinates around $x = \alpha(\xi) \in M$ and z^μ are local coordinates around $f(x) = \beta(\xi) \in N$.

If $k > \ell$, then ignoring all derivatives above the ℓ-th, yields the natural projection, π^k_ℓ of the k-jet bundle on the ℓ-jet bundle

$$\pi^k_\ell : J^k(M,N) \to J^\ell(M,N)$$
$$j^k_x f \mapsto j^\ell_x f. \tag{1.1.5}$$

In local coordinates this amounts to

$$\pi^k_\ell(x^a, z^\mu, z^\mu_a, \ldots, z^\mu_{a_1 \ldots a_k}) = (x^a, z^\mu, z^\mu_a, \ldots, z^\mu_{a_1 \ldots a_\ell}). \tag{1.1.6}$$

$J^o(M,N)$ may be identified with $M \times N$.

If $f \in C^\infty(M,N)$ then the <u>k-jet extension of f</u> is the map

$$j^k f : U \to J^k(M,N) \tag{1.1.7}$$

defined by

$$x \mapsto j^k_x f. \tag{1.1.7a}$$

In order to decide whether a map

$$\phi : M \to J^k(M,N) \tag{1.1.8}$$

is the k-jet extension of a function $f \in C^\infty(M,N)$ i.e.,

$$\phi(x) = j^k_x f \tag{1.1.9}$$

the <u>contact module Ω^k</u> over $C^\infty(J^k(M,N))$ is introduced as the module of 1-forms θ on $J^k(M,N)$ such that

$$(j^k f)^* \theta = 0 \tag{1.1.10}$$

for every function $f \in C^\infty(M,N)$.
In (1.1.10) $(j^k f)^* \theta$ is the <u>pull back</u> of the form θ to $U \subset M$ by the function $j^k f$.

In standard coordinates a basis for Ω^k is given by

$$\begin{aligned} \theta^\mu &= dz^\mu - z^\mu_c\, dx^c \\ \theta^\mu_a &= dz^\mu_a - z^\mu_{ac}\, dx^c \\ &\vdots \\ \theta^\mu_{a_1\ldots a_{k-1}} &= dz^\mu_{a_1\ldots a_{k-1}} - z^\mu_{a_1\ldots a_{k-1}c}\, dx^c. \end{aligned} \qquad (1.1.11)$$

The introduction of the module Ω^k is motivated by:
ϕ is the k-jet extension of a function $f \in C^\infty(M,N)$ iff $\phi^*(\Omega^k) = 0$.

For later use we define the total derivative vector fields on $J^k(M,N)$ by

$$D^{(k)}_a = \partial_a + z^\mu_a \partial_{z^\mu} + \ldots + z^\mu_{aa_1\ldots a_{k-1}} \partial_{z^\mu_{a_1\ldots a_{k-1}}} \qquad (1.1.12)$$

Total derivative vector fields commute

$$[D^{(k)}_a, D^{(k)}_b] = 0, \qquad (1.1.13)$$

where in (1.1.13) [,] denotes the Lie bracket of vector fields.
A system of partial differential equations of order k, for short a differential equation, can be described by functions

$$F^h : J^k(M,N) \to \mathbb{R} \quad (h=1,\ldots,c). \qquad (1.1.14)$$

A <u>differential equation</u> is a subset $\mathcal{Y} \subset J^k(M,N)$ being the zero set of

$$F : J^k(M,N) \to \mathbb{R}^c \qquad (1.1.15)$$

$$\xi \in \mathcal{Y} \mapsto 0. \qquad (1.1.15a)$$

A <u>solution of a differential equation</u> is a map $f \in C^\infty(M,N)$ such that

$$j^k_x f \in \mathcal{Y} \text{ for } x \in U \subset M. \qquad (1.1.16)$$

The <u>r-th prolongation</u> $\mathcal{Y}^r$ of a differential equation is defined as the zero set of

$$p^r F : J^{k+r}(M,N) \to \mathbb{R}^q.$$

where p^rF is defined by

$$\xi \in J^{k+r}(M,N) \to (F^h(\pi_k^{k+r}\xi),(D_a^{(k+r)}F^h)(\pi_{k+1}^{k+r}\xi),\ldots,(D_{a_1\ldots a_r}^{(k+r)}F^h)(\xi))$$

$$(q = c \sum_{s=0}^{r} \binom{\dim M + s-1}{s}). \tag{1.1.17}$$

In (1.1.17) $D_a^{(k+r)} F^h$ denotes the Lie derivative of the function F^h with respect to the total derivative vector field $D_a^{(k+r)}$; $D_a^{(k+r)}F^h,\ldots,D_{a_1\ldots a_r}^{(k+r)} F^h$ are the differential consequences of F^h.
We use the notation

$$D_{a_1\ldots a_r}^{(k+r)} = D_{a_1}^{(k+r)}\cdots D_{a_r}^{(k+r)}. \tag{1.1.17a}$$

In terms of differential forms a differential equation of order k can be described by a closed ideal I of differential forms defined on $J^k(M,N)$.
An ideal I of differential forms is closed if

$$dI \subset I. \tag{1.1.18}$$

Without loss of generality we can assume that a closed ideal I is generated by homogeneous forms $\alpha_1,\ldots,\alpha_m$ and their exterior derivatives $d\alpha_1,\ldots,d\alpha_m$

$$I = \langle\alpha_1,\ldots,\alpha_m,d\alpha_1,\ldots,d\alpha_m\rangle. \tag{1.1.19}$$

In the sequel we shall sometimes call a closed ideal I an exterior differential system, and we shall always assume an ideal I to be closed.
In most problems treated in chapters 3,4 we construct the ideal I of differential forms associated with a differential equation (1.1.15) in the following way:
start at the basis of contact 1-forms defined in (1.1.11) and solve each of

$$F^h(\xi) = 0 \quad (h= 1,\ldots,c)$$

for some $z^\mu_{a_1\ldots a_k}$ and substitute into (1.1.11); the ideal I is then generated by these new 1-forms and their exterior derivatives.
In effect we construct the restriction of the contact 1-forms to the differential equation $\mathcal{Y}$ (1.1.15).

A solution of an exterior differential system is a function $f \in C^\infty(M,N)$ such that

$$(j^k f)^* \, I = 0. \qquad (1.1.20)$$

Note that a solution defined by (1.1.20) of the exterior differential system (1.1.19) sketched above is a solution of $\mathcal{Y}$ (1.1.16) and vice versa.

The r-th prolongation of the ideal I, denoted $\overset{r}{D}\,I$ is defined as the ideal of differential forms on $J^{k+r}(M,N)$ generated by

$$\alpha_1,\ldots,\alpha_m, d\alpha_1,\ldots,d\alpha_m$$

$$D_a^{(k+r)}\alpha_1,\ldots,D_a^{(k+r)}d\alpha_m,\ldots,D_{a_1\ldots a_r}^{(k+r)}\alpha_1,\ldots,D_{a_1\ldots a_r}^{(k+r)}d\alpha_m. \qquad (1.1.21)$$

Actually in (1.1.21) we should write $(\pi_k^{k+r})^*\alpha_1,\ldots$ instead of $\alpha_1,\ldots$

A symmetry of an exterior differential system defined on $J^k(M,N)$ is a map

$$\phi : J^k(M,N) \to J^k(M,N) \qquad (1.1.22)$$

such that

$$\phi^* I \subset I. \qquad (1.1.23)$$

If $\phi_t : J^k(M,N) \to J^k(M,N)$ ($|t| < \varepsilon$) is a local 1-parameter group of symmetries, such that ϕ_0 is the identity map, then the generating vector field V of this 1-parameter group is called an infinitesimal symmetry of the exterior differential system, and the vector field V has to satisfy the condition

$$L_V \, I \subset I, \qquad (1.1.24)$$

where L_V denotes the Lie derivative with respect to the vector field V. The infinitesimal symmetries of an exterior differential system constitute a Lie algebra under the standard Lie bracket of vector fields.

Given a differential form β, then

$$L_V d\beta = dL_V\beta \qquad (1.1.25)$$

i.e., the Lie derivative with respect to a vector field V and the exterior derivative commute.
Relation (1.1.25) will be used in applications in the following way:
Let I be generated by $\alpha_1,\ldots,\alpha_m,d\alpha_1,\ldots,d\alpha_m$ (1.1.19), then if the vector field V satisfies

$$L_V\alpha_i \in I, \quad (i=1,\ldots,m) \qquad (1.1.26)$$

the condition

$$L_V d\alpha_i \in I \ (i=1,\ldots,m) \qquad (1.1.27)$$

is satisfied, due to (1.1.25).
So, if V satisfies (1.1.26), it satisfies (1.1.24) and is by consequence an infinitesimal symmetry of I.

We now state the following theorem about the structure of the components of an infinitesimal symmetry [1].

Theorem 1.1.1. (Bäcklund)

Let the vector field V defined on $J^k(M,N)$ be an infinitesimal symmetry of an exterior differential system I,

$$V = \xi^a\partial_{x^a} + \eta^\mu\partial_{z^\mu} + \eta^\mu_a\,\partial_{z^\mu_a} + \ldots + \eta^\mu_{a_1\ldots a_k}\,\partial_{z^\mu_{a_1\ldots a_k}} \qquad (1.1.28)$$

then there are two cases to be considered

case a : dim N > 1

the components of V (1.1.28) have the following structure

$$\begin{aligned}
\xi^a &= \xi^a(x^a,z^\mu) \\
\eta^\mu &= \eta^\mu(x^a,z^\mu) \\
\eta^\mu_a &= D^{(k)}_a(\eta^\mu) - z^\mu_b D^{(k)}_a(\xi^b) \\
&\vdots \\
\eta^\mu_{a_1\ldots a_k} &= D^{(k)}_{a_k}(\eta^\mu_{a_1\ldots a_{k-1}}) - z^\mu_{a_1\ldots a_{k-1}b} D^{(k)}_{a_k}(\xi^b)
\end{aligned} \tag{1.1.29}$$

i.e. the vector field V (1.1.28) generates a local group of point transformations $\overline{\phi_t} : J^0(M,N) \to J^0(M,N)$.

case b : dim $N = 1$ $(z^1 \equiv z)$
the most general situation is that there is a function

$$W = W(x^a,z,z_a) \tag{1.1.30}$$

such that the components of V (1.1.28) are given by

$$\xi^a = -\frac{\partial W}{\partial z_a} \quad , \eta = W - z_b \frac{\partial W}{\partial z_b} \quad , \eta_a = \frac{\partial W}{\partial x^a} + z_a \frac{\partial W}{\partial z} \tag{1.1.31}$$

while the other components of V are obtained in the same way as in case a (1.1.29), i.e., the vector field V (1.1.31) generates a local group of Lie contact transformations [1], $\bar{\phi}_t : J^1(M,N) \to J^1(M,N)$

The first idea to obtain a generalization of the notion of symmetry was to define them as transformations

$$\phi : J^{k+r}(M,N) \to J^{k+r}(M,N) \tag{1.1.32}$$

such that

$$\phi^*(D^r I) \subset (D^r I) \tag{1.1.33}$$

or for infinitesimal symmetries, where V is defined as a vector field on $J^{k+r}(M,N)$

$$L_V(D^r I) \subset (D^r I) \tag{1.1.34}$$

However, due to theorem 1.1.1 it was demonstrated [1], [34] that the condition (1.1.32) does not lead to a larger class of symmetries.
A generalization, which does lead to new classes of symmetries is pointed out in section 2 of this chapter.

Let the functional ℓ be defined by

$$\ell(z) = \int L(x^a, z^\mu, z^\mu_a) dx \tag{1.1.35}$$

where the Lagrangian L depends on x^a, z^μ, z^μ_a and let the differential equation (1.1.15) be just the Euler-Lagrange equation associated to (1.1.35), then

Theorem 1.1.2 (Noether)

If $\ell(z)$ is invariant under the infinitesimal symmetry V (1.1.28) of the differential equation (1.1.15), then $A = (A^a)$ defined by

$$A^a = (\eta^\mu - z^\mu_b \xi^b) \frac{\partial L}{\partial z^\mu_a} + L\xi^a \tag{1.1.36}$$

is a conserved vector i.e.,

$$D^{(k)}_a A^a = 0 \quad \text{on } \mathcal{Y} \tag{1.1.37}$$

or equivalenty

$$A^a \omega_a \tag{1.1.38}$$

is a conserved current associated to I i.e.,

$$d(A^a \omega_a) \in I. \tag{1.1.39}$$

In (1.1.38) (1.1.39) ω_a is defined by

$$\begin{aligned} \omega &= dx^1 \ldots dx^{\dim M} \\ \omega_a &= \partial_a \lrcorner\, \omega \quad (\lrcorner \text{ means contraction}) \end{aligned} \tag{1.1.40}$$

Similarity solutions are introduced according to [12].
Let I be an exterior differential system defined on $J^k(M,N)$ and let the vector field V be an infinitesimal symmetry of I (1.1.24).
We construct the ideal I' from I (1.1.21) in the following way

$$I' = \langle \alpha_1, \dots, d\alpha_m, V \lrcorner \alpha_1, \dots, V \lrcorner d\alpha_m \rangle. \tag{1.1.41}$$

The ideal I' is closed i.e., $dI' \subset I'$.
Moreover the vector field V is a Cauchy characteristic [9] of the ideal I' i.e.,

$$V \lrcorner I' \subset I' \tag{1.1.42}$$

and due to Cartan's theory on exterior differential systems we have a dimension reduction; for instance a partial differential equation in 2 independent variables reduces to an ordinary differential equation.

To demonstrate this technique we discuss
Example 1.1.1
The Heat equation is given by

$$u_t = u_{xx} \; (x^1 = x, x^2 = t, z^1 = z = u). \tag{1.1.43}$$

A closed ideal I of differential forms defined on $\mathbb{R}^7 = \{(x,t,u,u_x,u_t,u_{xt},u_{tt})\}$ is generated by

$$\begin{aligned} \alpha_1 &= du - u_x\, dx - u_t\, dt \\ \alpha_2 &= du_x - u_t\, dx - u_{xt}\, dt \\ \alpha_3 &= du_t - u_{xt} dx - u_{tt}\, dt \end{aligned} \tag{1.1.44}$$

and the exterior derivatives $d\alpha_1$, $d\alpha_2$, $d\alpha_3$.

The Lie algebra of infinitesimal symmetries is formed by [12]

$$\begin{aligned} &X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = u\partial_u, \quad X_4 = 2t\partial_x - xu\partial_u \\ &X_5 = x\partial_x + 2t\partial_t, \quad X_6 = xt\partial_x + t^2\partial_t + (-\tfrac{x^2}{4} - \tfrac{t}{2})\, u\, \partial_u \end{aligned} \tag{1.1.45}$$

$X_7 = g(x,t)\partial_u$, where $g(x,t)$ is a function satisfying (1.1.43).

In (1.1.45) we only write the ∂_x-, ∂_t-, ∂_u- components of the vector fields, the other components are defined by (1.1.29). A similarity solution associated with X_4 is obtained by the contraction of X_4 and α_1

$$X_4 \lrcorner \alpha_1 = -xu - 2tu_x. \tag{1.1.46}$$

It can be shown that the other conditions (1.1.41) lead to differential consequences of (1.1.46).
To construct a solution of (1.1.41) we have to solve

$$u_t = u_{xx} \tag{1.1.47a}$$
$$-xu - 2tu_x = 0 \tag{1.1.47b}$$

From (1.1.47b) we obtain

$$u(x,t) = H(t)\ e^{-\frac{x^2}{4t}}, \tag{1.1.48}$$

and substitution of (1.1.48) into (1.1.47a) results in an ordinary differential equation for the function H(t)

$$\frac{dH(t)}{dt} = -\frac{1}{2t} H(t).$$

from which we obtain the following well-known solution of the heat equation

$$u(x,t) = \frac{C}{\sqrt{t}}\ e^{-\frac{x^2}{4t}}. \tag{1.1.49}$$

□

An elegant application of infinitesimal symmetries in the study of nonlinear differential equations is given by the following

<u>Theorem 1.1.3</u> (Kumei & Bluman) [23]
A scalar n-th order nonlinear differential equation

$$F(x^a, z, z_a, \dots, z_{a_1 \dots a_n}) = 0 \quad x^a \in \mathbb{R}^m,\ z \in \mathbb{R} \tag{1.1.50}$$

is transformable by a 1 - 1 contact transformation to a linear differential equation if and only if the differential equation (1.1.50) admits an infinitesimal symmetry of the form

$$V = [\sigma(w)Z(\bar{X}^a(w))]\partial_z + \dots \tag{1.1.51}$$

where $w = (x^a, z, z_a)$

1°. $Z : \mathbb{R}^m \to \mathbb{R}$ $(X^a \mapsto Z(X^a))$ is an arbitrary solution of some n-th order linear differential equation

$$A(Z) = A(X^b)Z + A^a(X^b)Z_a + \dots + A^{a_1\dots a_n}(X^b)Z_{a_1\dots a_n} = 0 \tag{1.1.52}$$

2°. $\bar{X}^a : R^{2m+1} \to R^m$ $(w \mapsto \bar{X}^a(w))$ is a component of a Lie contact transformation

$$X^a = \bar{X}^a(w),\ Z = \bar{Z}(w),\ Z_a = \bar{Z}_a(w) \tag{1.1.53}$$

and

$$\begin{aligned} \partial_{z_a}\bar{Z}(w) &= \bar{Z}_b(w)\partial_{z_a}\bar{X}_b(w) \\ \sigma(w) &= (\partial_z\bar{Z}(w) - \bar{Z}_a(w)\partial_z\bar{X}^a(w))^{-1}. \end{aligned} \tag{1.1.54}$$

The transformation (1.1.53), (1.1.54) maps (1.1.50) to a linear differential equation

$$A(Z) - \Phi(X^a) = 0\ . \tag{1.1.55}$$

Example 1.1.2

We consider Thomas' equation [38]

$$z_{xy} + \alpha z_x + \beta z_y + \gamma z_x z_y = 0 \qquad (\alpha,\beta,\gamma \neq 0) \tag{1.1.56}$$

The Lie algebra of infinitesimal symmetries of (1.1.56) is generated by

$$\begin{aligned} &V_1 = \partial_x\ ,\ V_2 = \partial_y\ ,\ V_3 = \partial_z \\ &V_4 = \gamma x\partial_x - \gamma y\partial_y + (\alpha y - \beta x)\partial_z + \dots \\ &V_5 = e^{-\gamma z}\, v(x,y)\partial_z + \dots \end{aligned} \tag{1.1.57}$$

where v(x,y) has to satisfy the linear differential equation

$$v_{xy} + \alpha v_x + \beta v_y = 0 \tag{1.1.58}$$

First of all V_5 (1.1.57) is just the vector field V in Theorem 1.1.3, while the left hand side of (1.1.58) is just $A(Z)$ of (1.1.52).

The Lie contact transformation 1.1.53) is determined by

$$X^1 = x, \; X^2 = y \tag{1.1.59}$$

and Z has to satisfy (1.1.54), so

$$e^{-\gamma z} = (\partial_z \bar{Z})^{-1} \tag{1.1.60}$$

From equation (1.1.60) we obtain [23]

$$Z = \frac{e^{\gamma z}}{\gamma} - \phi(x,y) \tag{1.1.61}$$

In effect the Lie contact transformation (1.1.53) is in this situation just a Lie point transformation (1.1.59), (1.1.61).

From (1.1.59), (1.1.61) we derive

$$\begin{aligned} x &= X^1 \\ y &= X^2 \\ z &= \gamma^{-1} \log \gamma(Z + \phi) \end{aligned} \tag{1.1.62}$$

and

$$\begin{aligned} z_x &= \gamma^{-1}(Z + \phi)^{-1}(Z_{X^1} + \phi_{X^1}) \\ z_y &= \gamma^{-1}(Z + \phi)^{-1}(Z_{X^2} + \phi_{X^2}) \\ z_{xy} &= \gamma^{-1}(Z + \phi)^{-1}(Z_{X^1X^2} + \phi_{X^1X^2}) - \gamma^{-1}(Z + \phi)^{-2}(Z_{X^1} + \phi_{X^1})(Z_{X^2} + \phi_{X^2}) \end{aligned} \tag{1.1.63}$$

Substitution of (1.1.62) (1.1.63) into (1.1.56) results in

$$\gamma^{-1}(Z + \phi)^{-1}\{Z_{X^1X^2} + \alpha Z_{X^1} + \beta Z_{X^2} - \Phi(X^1,X^2)\} = 0 \tag{1.1.64}$$

where

$$\Phi(X^1,X^2) = -\phi_{X^1X^2} - \alpha\phi_{X^1} - \beta\phi_{X^2} \tag{1.1.64a}$$

Equation (1.1.64) is in agreement with equation (1.1.55) of Theorem 1.1.3.
If we take

$$\phi(x,y) = \text{constant} \tag{1.1.65}$$

then the transformation defined by (1.1.62), (1.1.63) transforms the differential equation (1.1.56) into

$$Z_{X^1X^2} + \alpha Z_{X^1} + \beta Z_{X^2} = 0 \tag{1.1.66}$$

which is just equation (1.1.58). □

1.2 Lie-Bäcklund transformations

In this section we introduce Lie-Bäcklund transformations or higher order infinitesimal symmetries according to [34].

In section 1 we noticed the impossibility to generalize the notion of infinitesimal symmetries to finite prolongations of exterior differential systems in order to obtain a more general setting, leading to new classes of infinitesimal symmetries.
The so called higher KdV-equations [21], [28] do not fit within the setting of finite prolongations.

The way out of this difficulty is offered by the introduction of the infinite jet bundle $J(M,N)$, which can be obtained as projective limit of $J^k(M,N)$.
Natural local coordinates on $J(M,N)$ are given by

$$x^a, z^\mu, z^\mu_a, \ldots, z^\mu_{a_1 \ldots a_k}, \ldots \quad . \tag{1.2.1}$$

Let π_k be the natural projection of $J(M,N)$ on $J^k(M,N)$.
A smooth function f is a function defined on $J(M,N)$ such that there is some k and some $f^k \in C^\infty(J^k(M,N))$ yielding

$$f = f^k \circ \pi_k, \tag{1.2.2}$$

and f is said to factor through $J^k(M,N)$; in effect f is only dependent on a finite number of variables. The class of smooth functions on $J(M,N)$ is denoted by $C^\infty(J(M,N))$.

A smooth vector field on $J(M,N)$ has a coordinate presentation

$$X = \xi^a \partial_a + \eta^\mu \partial_{z^\mu} + \ldots + \eta^\mu_{a_1 \ldots a_k} \partial_{z^\mu_{a_1 \ldots a_k}} + \ldots \tag{1.2.3}$$

where $\xi^a,\ \eta^\mu, \ldots, \eta^\mu_{a_1 \ldots a_k}, \ldots \in C^\infty(J(M,N))$.

Since every smooth function f factors through some $J^k(M,N)$ there are no problems concerning convergence in taking the Lie derivative

$$Xf = L_X f \tag{1.2.4}$$

of a smooth function f with respect to a vector field X (1.2.3) defined on J(M,N).

Total derivative vector fields on J(M,N) are given in natural local coordinates by

$$D_a = \partial_a + z^\mu_a \partial_{z^\mu} + \dots + z^\mu_{aa_1\dots a_k} \partial_{z^\mu_{a_1\dots a_k}} + \dots \tag{1.2.5}$$

A basis for the <u>infinite order contact module</u> Ω is given by the infinite set of contact 1-forms (c.f. (1.1.11)

$$\begin{aligned} \theta^\mu &= dz^\mu - z^\mu_b dx^b \\ &\vdots \\ \theta^\mu_{a_1\dots a_k} &= dz^\mu_{a_1\dots a_k} - z^\mu_{a_1\dots a_k b} dx^b \\ &\vdots \end{aligned} \tag{1.2.6}$$

Total derivative vector fields D_a (1.2.5) commute and annull the infinite order contact module Ω.

If the exterior differential system I is generated by the forms

$$\alpha_1, \dots, \alpha_m, d\alpha_1, \dots, d\alpha_m \tag{1.2.7}$$

then the <u>infinitely prolonged exterior differential system</u> $D^\infty I$ is generated by

$$\begin{array}{ccccc} \alpha_1, \dots, \alpha_m, & d\alpha_1, \dots, d\alpha_m & & & \\ D_a\alpha_1, & . & . & . & , D_a d\alpha_m \\ \vdots & & & & \vdots \\ D_{a_1\dots a_k}\alpha_1, & . & . & . & , D_{a_1\dots a_k} d\alpha_m \\ \vdots & & & & \vdots \end{array} \tag{1.2.8}$$

where in (1.2.8)

$$D_{a_1\dots a_k} = D_{a_1} D_{a_2} \dots D_{a_k}. \tag{1.2.8a}$$

Now, a Lie-Bäcklund transformation is a vector field V defined on J(M,N) such that

$$L_V(D^\infty I) \subset D^\infty I \ . \qquad (1.2.9)$$

Due to the construction of $D^\infty I$ (1.2.8) the total derivative vector fields D_a (1.2.5) satisfy (1.2.9) in an obvious way; in general the vector field Y defined by

$$Y = \xi^a D_a \qquad (1.2.10)$$

where $\xi^a \in C^\infty(J(M,N))$ satisfies (1.2.9).

Two vector fields X_1, X_2 defined on J(M,N) are equivalent if there are smooth functions $\xi^a \in C^\infty(J(M,N))$ such that

$$X_1 = X_2 + \xi^a D_a \qquad (1.2.11)$$

Combination of (1.2.3), (1.2.11) shows that we can restrict the study of Lie-Bäcklund transformations to vertical vector fields V i.e., those vector fields for which

$$(\pi_M)_* V = 0, \qquad (1.2.12)$$

where in (1.2.12) π_M is the natural projection of J(M,N) on M.

The Lie-Bäcklund transformations which are vertical vector fields can be shown to have the following coordinate representation [34]

$$V = \eta^\mu \partial_{z^\mu} + \eta^\mu_a \partial_{z^\mu_a} + \ldots + \eta^\mu_{a_1 \ldots a_k} \partial_{z^\mu_{a_1 \ldots a_k}} + \ldots \qquad (1.2.13)$$

where the functions $\eta^\mu_a, \ldots, \eta^\mu_{a_1 \ldots a_k}, \ldots \in C^\infty(J(M,N))$ are defined by

$$\begin{aligned} \eta^\mu_a &= D_a \eta^\mu \\ &\vdots \\ \eta^\mu_{a_1 \ldots a_k} &= D_{a_1 \ldots a_k} \eta^\mu \\ &\vdots \end{aligned} \qquad (1.2.14)$$

The functions η^{μ} (1.2.13) are called the generating functions of the vector field V and we shall use the notation

$$V = \eta^{\mu}\partial_{z^{\mu}} + \dots \tag{1.2.15}$$

instead of (1.2.13), (1.2.14).

The vector field V (1.2.13), (1.2.14) commutes with the total derivative vector fields D_a (1.2.5).

From these observations we conclude that it suffices to satisfy

$$L_V I \subset D^{\infty} I \tag{1.2.16}$$

instead of (1.2.9).
Finally, since the vector field V (1.2.15) is generated by smooth functions η^{μ}, i.e., dependent on a finite number of variables, condition (1.2.9), (1.2.16) reduces to the condition

$$L_V\, I \subset D^r I \qquad \text{for some } r. \tag{1.2.17}$$

Condition (1.2.17) will be used in the construction of Lie-Bäcklund transformations in applications in chapter 4.

We mention at this point that it is not possible in general to obtain a local 1-parameter group of transformations from a Lie-Bäcklund transformation.
Lie-Bäcklund transformations are used in the construction of higher order conserved currents and higher order conservation laws [22], [24], [34]. Moreover it is possible to use them in the construction of multisoliton solutions.
We shall shortly demonstrate this technique at the KdV-equation

$$u_t = uu_x + u_{xxx}. \tag{1.2.18}$$

The KdV-equation admits an infinite hierarchy of commuting Lie-Bäcklund transformations [22], [28].

$$X_1 = u_x \partial_u + \ldots$$
$$X_2 = (u_{xxx} + uu_x)\partial_u + \ldots \qquad (1.2.19)$$
$$X_3 = (u_{xxxxx} + \tfrac{5}{3} u_{xxx}u + \tfrac{10}{3} u_{xx}u + \tfrac{5}{6} u_x u^2)\ \partial_u + \ldots$$

The generating functions (1.2.15) of X_n (1.2.19) are obtained by the action of the Lenard recursion operator in the following way [21]

$$\eta_n = (D^2 + \tfrac{2}{3} u + \tfrac{1}{3} u_x D^{-1})\ \eta_{n-1} \qquad (1.2.20)$$

where in (1.2.20)

$$D = D_x$$
$$D^{-1} f = \int_{-\infty}^{x} f dx$$

In (1.2.19) all t-derivatives are eliminated using (1.2.18) and its differential consequences (1.1.17)
The Lie-Bäcklund transformation X_1 is equivalent to the infinitesimal symmetry ∂_x (1.2.11) [12], while X_2 is equivalent to ∂_t .

Contraction of the vector field $X_3 + pX_2 + qX_1$ (1.2.19) with the contact form

$$\theta = du - u_x dx - u_t dt$$

leads to the similarity condition

$$u_{xxxxx} + \tfrac{5}{3} u_{xxx}u + \tfrac{10}{3} u_{xx}u_x + \tfrac{5}{6} u_x u^2 + p(u_{xxx} + u_x u) + qu_x = 0 \qquad (1.2.21)$$

In [25], Lax demonstrated that solving the system (1.2.18), (1.2.21) results in the two soliton solution, the solution of the KdV-equation consisting of two interacting solitary waves travelling at different speeds.

1.3. Nonlocal symmetries

To indicate the way in which the notion of infinitesimal symmetry might be generalized, let us consider the following example [20].

Example 1.3.1

Let $M = \mathbb{R}^2$ with local coordinates $x^1 = x$; $x^2 = t$ and $N = \mathbb{R}$ with local coordinates $z = z^1 = u$.
The well-known Burgers' equation is given by

$$u_t = uu_x + u_{xx} \tag{1.3.1}$$

An exterior differential system I defined on Y associated to (1.3.1) is generated by 3 1-forms

$$\begin{aligned} \alpha_1 &= du \quad -u_x\, dx - u_t\, dt \\ \alpha_2 &= du_x - (u_t - uu_x)dx - u_{xt}dt \\ \alpha_3 &= du_t \quad -u_{xt}\, dx - u_{tt}dt \end{aligned} \tag{1.3.2}$$

and their exterior derivatives $d\alpha_1$, $d\alpha_2$, $d\alpha_3$.
The Lie algebra of infinitesimal symmetries of the ideal I is easy to derive, is 5-dimensional and generated by

$$\begin{aligned} V_1 &= \partial_x & V_4 &= t\partial_x - \partial_u \\ V_2 &= \partial_t & V_5 &= xt\partial_x + t^2\partial_t - (x+ut)\partial_u \\ V_3 &= -x\partial_x - 2t\partial_t + u\partial_u \end{aligned} \tag{1.3.3}$$

In (1.3.3) only the ∂_x-, ∂_t-, ∂_u- components are given; the other components are obtained from (1.3.3) using (1.1.29). The vector fields $V_1,\dots,V_5$ (1.3.3) are equivalent (1.2.11) to the vertical vector fields whose generating functions are GG(1),...,GG(5) (4.1.18b).
Now notice that

$$\beta = 2udx + (u^2+2u_x)dt \tag{1.3.4}$$

is a conserved current to I, i.e.,

$$d\beta = 2dudx + 2ududt + 2du_x dt$$
$$= (-2dx - 2udt)\,\alpha_1 - 2dt\alpha_2 \in I. \tag{1.3.5}$$

We construct the ideal I' by prolongation of the ideal I with the potential form

i.e., $$\alpha_4 = dp - 2udx - (u^2 + 2u_x)dt \tag{1.3.6}$$
$$I' = \langle \alpha_1, \alpha_2, \alpha_3, d\alpha_1, d\alpha_2, d\alpha_3, \alpha_4 \rangle. \tag{1.3.7}$$

2-dimensional solution manifolds of (1.3.7), where x,t are chosen as independent variables have to satisfy the differential equation

$$\begin{aligned} u_t &= uu_x + u_{xx} \\ p_x &= 2u \\ p_t &= u^2 + 2u_x. \end{aligned} \tag{1.3.8}$$

Computation of the Lie algebra of infinitesimal symmetries of I' (1.3.7) leads to the remarkable result:
The Lie algebra is 7-dimensional and generated by

$$\begin{aligned} &\tilde{V}_1 = \partial_x && \tilde{V}_4 = t\partial_x - \partial_u - 2x\partial_p \\ &\tilde{V}_2 = \partial_t && \tilde{V}_5 = xt\partial_x + t^2\partial_t - (x+ut)\partial_u - (x^2+2t)\partial_p \\ &\tilde{V}_3 = -x\partial_x - 2t\partial_t + u\partial_u && \tilde{V}_6 = \partial_p \end{aligned} \tag{1.3.9}$$

and

$$\tilde{V}_7 = (-2g_x(x,t) + g(x,t)u)\, e^{-\frac{p}{4}}\, \partial_u - 4g(x,t)e^{-\frac{p}{4}}\, \partial_p, \tag{1.3.9a}$$

where g(x,t) is a arbitrary solution of the Heat equation

$$g_t = g_{xx}. \tag{1.3.9b}$$

In (1.3.9) only ∂_x-, ∂_t-, ∂_u-, ∂_p-components are given.

Since formally $p = \int_{-\infty}^{x} 2udx$, due to (1.3.8), p is called a nonlocal variable. In (1.3.9) the local components of the vector fields $\tilde{V}_j (j=1,\dots,7)$ are obtained from the conditions

$$L_V \alpha_i \in I' \quad (i=1,\ldots,3) \tag{1.3.10}$$

while the nonlocal ∂_p-component is obtained by using the result of (1.3.10) in

$$L_V \alpha_4 \in I'. \tag{1.3.11}$$

If we now contract the vector field and the 1-form α_1 we obtain an additional condition to (1.3.1) i.e.,

$$-2g_x + g\, u = 0 \tag{1.3.12}$$

or equivalently

$$u = \frac{2g_x}{g}. \tag{1.3.12a}$$

Substitution of (1.3.12a), (1.3.9b) into (1.3.1) shows that (1.3.1) is automatically satisfied.

(1.3.12), (1.3.9b) is the well-known Cole-Hopf transformation, with associates to any solution of the Heat equation (1.3.9b) a solution (1.3.12) of Burgers' equation.

We stress at this point that (1.3.12) is not obtained from (1.3.7) by the notion of similarity solutions (1.1.22 - 1.1.24).

So the notion of infinitesimal symmetries of exterior differential systems can be generalized to incorporate nonlocal variables, leading to new and interesting classes of nonlocal infinitesimal symmetries.

In chapter 4 section 4 we indicate how the notion of Lie-Bäcklund transformations introduced in section 2 can be generalized to include nonlocal variables. This generalization, based on the prolongations of $D^\infty I$ by potential forms, leads to conditions equivalent to those obtained by Krasilshchik & Vinogradov using the prolongation of the total derivative vector fields D_a (1.2.5).

Chapter 2

DESCRIPTION OF SOFTWARE TO COMPUTE INFINITESIMAL SYMMETRIES OF EXTERIOR DIFFERENTIAL SYSTEMS

2.0 Introduction

The huge amount of essential standard computations necessary to obtain the Lie algebra of infinitesimal symmetries of an exterior differential system leads in a natural way to the question: how to do these computations with the help of a computer?

In this chapter we shall describe some software to carry through these computations on a computer system.

We use REDUCE [13], a symbolic language based on LISP [8].

In section 1 we shall indicate what kind of problems one comes across mathematically, and we shall mention the procedures which are constructed to execute these computations on a computer system.

The reader is not assumed to have knowledge of REDUCE or LISP in order to understand section 1.

In section 2 we shall give a general description of the procedures, suited for a user of the developed software.

In section 3 we give detailed information about technicalities of the construction of the procedures. We make references to the source code which is given in the appendix at the end of this tract.

Finally in section 4 we reconstruct a complete computer session of the computation of the infinitesimal symmetries of Burgers' equation. The aim of this section is to demonstrate the way how a problem may be solved.

Recently Steinberg [37] constructed software suited for the symbolic language MAXSYMA. Schwartz [33] constructed a software package quite similar to ours, but based on doing computations completely automatic; however it is our experience that due to this, in some sense, over-automation, the system is very time-consuming and leads to expression-swell.

2.1 Basic ideas

To compute infinitesimal symmetries we start with a closed ideal I of differential forms, defined on an n-dimensional space $\mathbb{R}^n = \{(x(1),\ldots,x(n))\}$ and generated by homogeneous differential forms $\alpha(i)$ $(i=1,\ldots,m)$,

$$I = \langle\alpha(1),\ldots,\alpha(m)\rangle. \tag{2.1.1}$$

The vector field V, defined by

$$V = \sum_{i=1}^{n} F(i)\partial_{x(i)} \tag{2.1.2}$$

where F(i) $(i=1,\ldots,n)$ are functions defined on $\mathbb{R}^n$, is an infinitesimal symmetry of (2.1.1) if V satisfies the condition (1.1.24)

$$L_V\, I \subset I. \tag{2.1.3}$$

Equation (2.1.3) is equivalent to

$$L_V\alpha(k) + \sum_{j=1}^{m} \gamma(k,j)\alpha(j) = 0 \quad (k=1,\ldots,m) \tag{2.1.3a}$$

where $\gamma(k,j)$ are suitable differential forms.

Elimination of the coefficients of $\gamma(k,j)$ $(k,j=1,\ldots,m)$ from these conditions (2.1.3) results in an overdetermined, linear and homogeneous system of partial differential equations for the functions F(i) $(i=1,\ldots,n)$, the components of the vector field V. The procedure INFSYM generates this overdetermined system of partial differential equations.

The nature of these overdetermined systems obtained from condition (2.1.3) is in many practical situations very specific [12], [16], [17], and one comes across very special types of partial differential equations which have to be solved.

The left hand sides of these partial differential equations are stored in REDUCE as values of an operator VER, so the values VER(1),...,VER(TOTAL) are just these left hand sides. A solution of (2.1.3) is a set of n functions F(i) (i=1,...,n) such that the evaluations of VER(1),...,VER(TOTAL) become zero.

The functions F(i) are given in REDUCE without explicit occurrence of their variables.
They are stored as values in an association list DEPL!*.
By F(*) we denote some function : F(index).
By DF(F(*),*) we denote some derivative of a function F(index) with respect to a list of variables : DF(F(index), list of variables).

We assume the partial differential equations to be of the following structure

<u>function</u> : F(*)
<u>derivative</u> : DF(F(*),*)
<u>coefficient</u> : an algebraic expression not containing a function F(*) or a DF(F(*),*)
<u>term</u> : <u>coefficient</u> * <u>function</u> | <u>coefficient</u> * <u>derivative</u>
<u>equation</u> : <u>term</u> | <u>term</u> + <u>equation</u>.

We shall now describe the most frequently occurring cases, which arise in practical situations. The procedure FINES searches for such situations and constructs the general solution to such equations.
The procedures ONETERMSOL, SPLIT, described in Gragert P.K.H., Kersten P.H.M. & Martini R.,[11] are just special situations, dealt with by the procedure FINES.

In examples we shall use a REDUCE-like notation; by VER(◦) we mean one of the VER(1),...,VER(TOTAL).

<u>CASE A</u>

A partial differential equation is of polynomial type in one (or more) of the variables, the functions F(*) appearing is this equation are independent of this variable; by consequence, each of the coefficients of the polynomial has to be zero, and the partial differential equation decomposes into some new and smaller equations.

Example 2.1.1

The partial differential equation is

$$VER(\cdot) := X(1)^2 * DF(F(1),X(2)) + X(1)*F(2) \tag{2.1.4}$$

where in (2.1.4) F(1), F(2) are functions independent of X(1).
By consequence the coefficients of the polynomial in X(1) have to be zero, i.e., DF(F(1),X(2)) and F(2).
So (2.1.4) is equivalent to

$$VER(\cdot) := DF(F(1),X(2)) \tag{2.1.5}$$
$$VER(\cdot) := F(2).$$

□

CASE B

The partial differential equation VER(•) represents some derivative of a function F( ).

In general

$$VER(\cdot) := DF(F(\ ),\ X(i_1),k_1,\ldots,X(i_r),k_r), \tag{2.1.6}$$

a mixed $(k_1+\ldots+k_r)$-th order derivative, while the k_i's are not present in case $k_i = 1$.
The general solution of (2.1.6) is

$$F(\ ) := \sum_{s=1}^{r} \sum_{t=0}^{k_s-1} F(i_{s,t}) * X(i_s)^t \tag{2.1.7}$$

whereas in (2.1.7) $F(i_{s,t})$ depends on the same variables as F( ), except $X(i_s)$ $(t=0,\ldots,k_s-1,\ s=1,\ldots,r)$.

Example 2.1.2

$$VER(\cdot) := DF(F(1),X(1),X(2)). \tag{2.1.8}$$

The general solution to this equation is given by

$$F(1) := F(2) + F(3), \tag{2.1.9}$$

where F(2) depends on the same variables as F(1), except X(1), while F(3) depends on the same variables as F(1), except X(2).

□

CASE C

The partial differential equation VER(•) contains a function F( ) depending on all variables present as arguments of some other function(s) F(*), occurring in this equation, whereas there is no derivative of the function F( ) present in the equation. The partial differential equation can be solved for this function F( ).

Example 2.1.3

The equation is

```
VER(•):= X(1)*F(1) + X(2)*DF(F(2),X(1))                    (2.1.10)
```

where in (2.1.10), F(1), F(2) are dependent on X(1), X(2), X(3).

The solution is

```
F(1):= (-X(2)*DF(F(2),X(1)))/X(1)                          (2.1.11)
```

□

CASE D

In the partial differential equation there is a derivative of a function F( ) with respect to variables which are not present as argument of any other function F(*), while the coefficient of F( ) is a number. By the assumption that X(1),...,X(n) appear as polynomials, the partial differential equation can be integrated.

Example 2.1.4

Let the partial differential equation be given by

```
VER(•):= DF(F(1),X(3)) + X(2) * F(2)                       (2.1.12)
```

where

F(1) depends on X(1), X(2), X(3), (2.1.13)
F(2) depends on X(1), X(2).

The solution to (2.1.12) is

F(1):= - X(2)*X(3)*F(2) + F(3) (2.1.14)

whereas F(3) depends on X(1), X(2) and is independent of X(3). □

CASE E

In the partial differential equation a specific variable X(i) is present just once as an argument of some function F(*). By appropriate differentiation one might arrive at a simple equation, which can be solved. Evaluation of the original equation can result in an equation which can be solved.

Example 2.1.5

VER(•):= X(2)*DF(F(1),X(2),X(3)) + X(3) * F(2) (2.1.15)

where

F(1) depends on X(1), X(2), X(3),
F(2) depends on X(1), X(2).

Differentiation, with respect to X(3) twice results in

VER(•):= X(2)*DF(F(1),X(2),X(3),3) (2.1.16)

The solution to (2.1.16) is (CASE B)

F(1):= F(3) * $X(3)^2$ + F(4) * X(3) + F(5) + F(6).

where

F(3), F(4), F(5) are dependent on X(1), X(2) (2.1.17)
F(6) depends on X(1), X(3).

Evaluation of (2.1.15) leads to

```
VER(•):= 2*X(2)*X(3)*DF(F(3),X(2)) + X(2)*DF(F(4),X(2) + X(3)*F(2).   (2.1.18)
```

Due to CASE A, FINES constructs two new equations

```
    VER(•):= 2*X(2)*DF(F(3),X(2)) + F(2)                              (2.1.19a)
    VER(•):= X(2)*DF(F(4),X(2))                                       (2.1.19b)
```

The effect of FINES applied to (2.1.15) will be (2.1.17), (2.1.19). □

The procedure FINES is then again useful to handle equations (2.1.19a,b). This last step is not executed automatically.

In many practical problems one is able to solve the overdetermined system of partial differential equations (2.1.3) using the methods described in CASE A,B,C,D,E, and some additional considerations due to the nature of the specific problem at hand.

The final result is a set of n expressions for the functions F(1),...,F(n), the components of the vector field V (2.1.2).
In these expressions there are several integration constants F(*) due to CASE B,D,E and even sometimes free functions [11] or functions which have to satisfy a genuine differential equation [12]. From the representation of this general
Vector Field V the procedure VFGEN constructs the set of GENerators of the Lie algebra of infinitesimal symmetries.

2.2 General description of the procedures

A general description of the procedures, used in the construction of the Lie algebra of inifinitesimal symmetries of exterior differential systems is given. The source code of the procedures is at the end of this thesis.
A description of LIEDF, INFSYM, GEFORM, FINES, FSOLV, ELIM, VFGEN is given.

In the construction of the overdetermined system of partial differential equations which has to be satisfied by the components of the vector field V in order to be an infinitesimal symmetry, we use the package, constructed by Gragert [10]. This package is used to carry through computations of differential geometric nature.
We list below some global data required by this package.

D!@DIF: the dimension of the vector space
OPERATOR VNAT : the coordinates of the vector space
OPERATOR UIT : to describe basis forms, e.g. dx ≡ UIT(1).
(UIT product is the Dutch word for wedgeproduct)

Procedures described in [10] and called within the procedure bodies of this section are EXDF, IP, NORMDIF, MULFORM, !@BEVATOP, OPCOEFF, OPL.

Moreover, throughout this section we assume the following global data to exist
DEPL!*, FPKTEL, TOTAL, LIOV
OPERATOR VER ; OPERATOR F.

TOTAL : the number of partial differential equations describing the conditions on the components of the vector field V.

FPKTEL: the total number of functions introduced in solving the overdetermined system.

DEPL!*: an association list available in REDUCE to store the arguments of functions.

LIOV : the name of the key to an element of the association list DEPL!* where all the independent variables of the overdetermined system are stored.

VER,F : the partial differential equations for the functions F(I) (I=1,...,D!@DIF i.e., the components of the vector field V) are stored as values of the operator VER : VER(1),...,VER(TOTAL).

Example 2.2.1

The value of DEPL!* is

(((F 1) (X 1) (X 2)) (LIOV (X 1) (X 2) (X 3))),

if there is only one function F(1) dependent on X(1),X(2), while there are X(1), X(2), X(3) as variables in the system.

□

LIEDF(*1,*2)

The procedure computes the Lie derivative of a differential form with respect to a vector field.

parameters:

*1 : the vector field

*2 : the differential form.

The procedure body is just the definition of the Lie derivative

$$L_V\ \alpha = V\lrcorner d\alpha + d(V\lrcorner\alpha).$$

INFSYM(*1,*2,*3)

The procedure constructs the overdetermined system of partial differential equations for the components of the vector field V.

parameters:

*1 : the number of the first differential form ALFA(*1) on which condition (2.1.3) is applied.

*2 : the number of the last differential form ALFA(*2) on which condition (2.1.3) is applied.
Condition (2.1.3) is applied to ALFA(*1),...,ALFA(*2)

*3 : the total number of differential forms ALFA(1),...,ALFA(*3) generating the exterior differential system.

global data:

OPERATOR ALFA,

ALFA(1),...,ALFA(*3),

F(1),...,F(D!@DIF).

procedure calls:

NORMDIF, MULFORM, OPCOEFF, !@BEVATOP, OPL, CLEARKVALUE, LIEDF, GEFORM.

result:

The result is the number of partial differential equations constructed by the procedure.

effect:

The overdetermined system of partial differential equations associated to the condition

$$L_V\ \alpha(i) \in I, \ (i=*1,\dots,*2)$$

are stored as values of VER(1),...,VER(TOTAL).

side effects:

1^o Message : THERE EXIST N EQUATIONS FOR ALFA (M)

2^o In case there are denominators introduced in the construction of the overdetermined system, there will be a message:
CREATION OF DENOMINATOR.

GEFORM(*1,*2)

The procedure constructs a GEneral differential FORM of order less than 4, with unspecified coefficients.

parameters:

*1 : an index to distinguish general differential forms

*2 : the order of the differential form ($\le$ 3).

global data :

OPERATOR CO: used to denote the unspecified coefficients of the differental form.

result :

The result is the required differential form of order *2.

FINES(*1)

The construction of the general solution of an equation VER(*1) or construction of an equivalent system.

parameter:

*1 : the number of the partial differential equation, VER(*1) to which the procedure is applied.

<u>global data</u>:
FIDEPT,
ARRAY FIPKA(20),
ARRAY FIPKA1 (the number of variables in the system).

FIDEPT: a global whose value avoids a recursion of the procedure FINES of depth greater than 2.

FIPKA(20) : an array to construct the coefficients of a polynomial (highest power ≤ 20).

FIPKA1(..): an array whose elements contain information about the number of occurences of a variable as argument of functions F(*) in VER(*1).

Note: the dimension 20 for the array FIPKA, turned out to be suitable for practical purposes. A user can redefine this array himself taking a dimension > 20.

<u>procedure calls</u> :
OPCOEFF, STRPOLY, FSOLV, CLEARKVALUE.

<u>effects</u>:
The construction of the general solution to the partial differential equation, or the construction of an equivalent system.

<u>side effect</u>:
there will be messages about actions and searches of the procedure:

```
CASE A : VER(*1) BREAKS INTO VER(•),...,VER(•) BY:
         (...).
CASE B : HOMOGENEOUS INTEGRATION OF : (DF(F *),*)
CASE C : VER(*1) IS SOLVED FOR : (F *)
CASE D : INHOMOGENEOUS INTEGRATION OF : (DF(F*),*)
CASE E : SEARCH FOR A DIFFERENTIATION
         ( )
         TOTAL :=
           :
         ELIMINATION OF VARIABLE(S) : FINES
           :
```

If the search in CASE D (section 1) is not successful there will be one of the following messages:

1^o : MORE THAN ONE MAXIMAL DF(F(*),*),
in case there are two or more derivatives having all variables present in the equation as their argument.

2^o : COEFFICIENT OF THE DERIVATIVE IS NOT A NUMBER.

In these situations no further action is taken.

Note: a DF(F(*),*) is called MAXIMAL in case F(*) depends on all the variables present in the partial differential equation VER(*1)

FSOLV(*1,*2)

the procedure solves an equation VER(*1) for a function F(*2).

parameters :

*1 : The number of the partial differential equations VER(*1).

*2 : The number of the function F(*2).

procedure call :

OPL

result :

NIL

effect :

The solution of the partial differential equation.

ELIM (*1,*2)

The purpose of the procedure is to replace the sum of two functions which occur in the system; they often arise from integration constants.

parameters:

*1 : the index of the function F(*1) which has to be eliminated

*2 : the index of the other function.

result :

NIL

effect : if allowed then F(*1) is replaced by

-F(*2) + F(**)   (F(**) : a new function)

side effect:

in case the elimination is not allowed a message :

WRONG ELIMINATION.

In this situation the elimination is not carried through.

VFGEN (*1)

The procedure decomposes the general solution of the overdetermined system into the generating vector fields of the Lie algebra of infinitesimal symmetries.

parameters :

*1 : The dimension of the vector fields to be constructed.

global data :

VFPKTEL	: a counter for vector fields
VFPKTRL	: a list of new variables
OPERATOR D	: the components of the vector fields
OPERATOR VF	: the name of the Vector Fields
OPERATOR VFV	: the name of an auxiliary vector field.

2.3 Detailed description of the procedures

A detailed description of the procedures INFSYM, FINES, FSOLV, ELIM, VFGEN will be given.
In the procedure body of FINES there is a procedure call STRPOLY (*1,*2), a procedure that constructs coeffcients of *1 considered as a multivariate polynomial in the variables of *2.
This procedure is almost the same as the procedure MLIETAB [10] and for that reason we shall not describe it here; the procedure MLIETAB was just adapted to our situation.

In this section, by "%C..." we shall refer to marks in the source code of the procedures.

The procedure bodies of the procedures LIEDF(*1,*2) and GEFORM(*1,*2) are obvious.

INFSYM :
(source code on p.143)
%C1 : declaration and initialization of variables
%C2 : reconstruction of the differential forms ALFA(I) [10] into canonical form
%C3 : construction of the vector field VEC
%C4 : computation of the order RA(JJ) of the differential forms ALFA(JJ)
%C5 : the condition $L_V I \subset I$
%C5.1 : construction of $L_V\alpha(i) + \gamma(i,j)\ \alpha(j)$
%C5.2 : computation of the coefficients of the basis forms
%C5.3 : elimination of the coefficients CO of the general forms $\gamma(i,j)$, in case they are multiplied by $\pm$ 1
%C5.4 : elimination of the coefficients CO of the general forms $\gamma(i,j)$, in case they are multiplied by coefficients not equal to $\pm$ 1; or introduction of a new partial differential equation, VER(*), for the functions F(I)

FINES:

(source code on p.146)

%CO.1 : declaration and initialization of variables

%CO.2 : IF the entry is not correct
THEN
1° : message : OUT OF RANGE 1,...,TOTAL
2° : GAVERDER:= NIL (i.e., no further action will be taken)

%CO.3 : If the equation is satisfied automatically by results, obtained previously
THEN 1° : message VER(*) IS ZERO
2° : GAVERDER:= NIL

%CO.4 : decomposition of the numerator of the equation into DF(F(*),*)-terms and their coefficients

%CO.5 : IF there is only one derivative
AND nothing else
THEN

%C1 : HOMOGENEOUS INTEGRATION (CASE B of section 2)

%CO.6 : decomposition of the part of the numerator not containing derivatives, into F(*)-terms and their coefficients

%CO.7 : construction of
WELAANWARG : the general list of variables
NIETAANWARG : the list of variables not present as argument of a F(*)

%CO.8 : determination of the coefficients of a polynomial expansion with respect to the elements of NIETAANWARG.

%CO.9 : IF the number of coefficients > 1
THEN

%CO.10 : 1° Message: VER(*)BREAKS INTO VER( ),...,VER( )
2° addition of new equations (CASE A, of section 2)
3° GAVERDER:= NIL

%C2 : SEARCH FOR A FUNCTION F(*), such that the equation can be solved for this function (CASE C of section 2)

%C3 : SEARCH FOR INHOMOGENEOUS INTEGRATION; i.e. search for a function F(*) having all variables, present in any other F(*)- or DF(F(*),*)-term as argument, as its arguments. Moreover the differentiation of this maximal F(*), must be with respect to variables not present elsewhere as argument of a function. (CASE D of section 2)

%C4 : SEARCH FOR A DIFFERENTIATION (CASE E, of section 2)

%C1 HOMOGENEOUS INTEGRATION in more detail:

%C1.1 : initialization of variables

DS0 : DF(F(*),*)

DS1 : F(*)

DS2 : ((F*)...)

DS3 : the list of arguments of F(*)

DS4 : the list of differentiation in DF(F(*),*)

DS8 : the list of terms

message : HOMOGENEOUS INTEGRATION OF : DF(F(..),...)

%C1.2 : CONSTRUCTION OF TERMS

$$\sum_{s=1}^{r} \sum_{t=0}^{k_s -1} F(i_{s,t}) * x(i_s)^t \qquad (2.1.7)$$

%C1.2.1: DS5 : $x(i_s)$

DS7 : $(x(i_1),...,\widehat{x(i_s)},...,x(i_r))$ ($\widehat{x(i_s)}$ means deletion)

%C1.2.2: IF $k_s > 1$

THEN construction of the terms

$F(i_{s,t}) * x(i_s)^t$ $(1 \leq t \leq k_s -1)$,

adjustment of DS8

%C1.2.3: construction of

$F(i_{s,0})$

%C1.3 : final construction of the solution and assignment to F(*)

GAVERDER:= NIL

%C2 : SEARCH FOR A FUNCTION F(*) in more detail:

%C2.1 : WELAANWARG : the list of variables present as an argument of some function F(*)

TOEGEDIFVAR: list of variables not present as arguments of some function F(*)

%C2.2 : OS1: the list of F(*) which depend on all arguments (WELAANWARG) and a derivative DF(F(*),*) of that function F(*) is present in the equation (it is not allowed to solve the equation for such F(*)).

%C2.3 : IF there is a function having all elements of WELAANWARG as argument

%C2.4 : THEN
IF the function is not on the list OS1
%C2.5 : THEN 1e message : VER(..) IS SOLVED FOR : F(..)
2e solve the equation for this function
3e GAVERDER:= NIL
%C2.6 : ELSE
adjustment of TOEGEDIFVAR

%C3 SEARCH FOR INHOMOGENEOUS INTEGRATION in more detail:
%C3.1 : initialization of variables
%C3.2 : IS3 : the F(*) in a DF(F(*),*)-term
IS4 : the list of arguments of this F(*)
%C3.3 : IF IS4 = WELAANWARG (the list of all arguments in the equation)
THEN
%C3.4 : IF IS2 = 0 (there is not yet a maximal DF(F(*),*)
THEN IS5: the index to retrieve the maximal DF(F(*)*) in PKDFC(0,*)
ELSE 1e : message : MORE THAN ONE MAXIMAL DF(F(*),*)
2e : GAVERDER:= NIL
%C3.5 : ELSE
adjustment of TOEGEDIFVAR
%C3.6 : IF IS5 THEN
%C3.7 : Check whether the list of differentiations (IS6) is appropriate to integrate the DF(F(*),*)-term (differentiation with respect to arguments which are on TOEGEDIFVAR)
%C3.8 : IF the coefficient of the DF(F(*),*)-term is a number
THEN
%C3.9 : construction of the inhomogeneous part of the solution
%C3.9.1: IS8 : the inhomogeneous term
%C3.9.2: integration of the inhomogeneous term with respect to the elements of differentiation of the DF(F(*),*)-term, using the procedure COEFF, under the assumption of polynomial structure
%C3.10: construction of the homogeneous part of the solution (this part is just the same as HOMOGENEOUS INTEGRATION)
GAVERDER:= NIL
%C3.11: ELSE(i.e., in case the coefficient of the DF(F(*),*)-term is not a number)

1° mesage: THE COEFFICIENT OF THE DERIVATIVE IS NOT A NUMBER

2° GAVERDER:= NIL (the intelligence of the user might be of interest)

%C4 SEARCH FOR A DIFFERENTIATION in more detail:

message: SEARCH FOR A DIFFERENTIATION

%C4.1 : initialization of variables

%C4.2 : counting of the variables present as argument of a F(*) in a DF(F(*),*)-term

DD6: the list of arguments of a particular DF(F(*),*)-term

DD9: the number of a variable of the list DD6

%C4.3 : FIPKA1(DD9): the number of occurrences of the variable (X DD9) as an argument in a DF(F*),*)-term

%C4.4 : counting of the variables present in a F(*)-term

%C4.5 : RESMET: the number of functions F(*) being present in the system at this stage

%C4.6 : IF there is a variable that occurs once as argument of a F(*) or DF(F(*),*)-term

THEN

%C4.7 : message: (X *)

%C4.8 : DD8 : the highest power of the variable (X *) occurring in the coefficients of the F(*)- or DF(F(*),*)-terms

VER(TOTAL +1): differentiation of the equation (DD8 + 1)-times with respect to (X *)

%C4.9 : action of the procedure FINES on the new equation

%C4.10 : adjustment of FIPKA1(*)

%C4.11 : IF new function F(*) has been introduced

THEN 1° message : ELIMINATION OF VARIABLE(S) : FINES

2° action of FINES

FSOLV:

(source code on p.144)

%C1 : solving the equation by the action of OPL

%C2 : VER(*):= 0

%C3 : adjustment of DEPL!*

ELIM:

(source code on p. 146)

%C1 : declaration and initialization of variables
L1 : the list of (F *1) and its arguments
L2 : the list of (F *2) and its arguments

%C2 : checking whether all the arguments of (F *2) are argument of (F *1); if not then there is a message

%C3 : If the elimination is allowed

%C4 : THEN the ELIMINATION PROCESS is carried cout

%C4 : ELIMINATION PROCESS in more detail:

%4.1 : L3 : the list of arguments of F(*1)
L4 : construction of a new function F(**)
L5 : - F(*2) + F(**)

%C4.2 : F(*1) := - F(*2) + F(**)

VFGEN:

(source code on p.145)

%C1 : declaration and initialization of variables
FLY : the list of ((F *)...) on DEPL!*

%C2 : IF there is a list of new variables
AND the number of elements is equal to the dimension of the vector field
THEN L1 := list of new variables
ELSE L1 := list of old variables

%C3 : construction of the vector fields

%C3.1 : IF the (F *) on the list FLY is a constant
THEN take F(*) equal to 1

%C3.2 : construction of the vector field associated to F(*);
deletion of the result in case it is a zero vector field

%C4 : IF there is an appropriate list of new variables
THEN reconstruction of DEPL!*

%C5 : introduction of ORDER and FACTOR for a nice printing of the result.
!*A2K is a REDUCE function that checks if the argument is a kernel and makes the reference unique.
message: CREATION OF A TOTAL NUMBER OF ... VECTOR FIELDS

2.4. A complete computer session

The purpose of this section is to demonstrate how the procedures described in section 1,2 can be used to compute infinitesimal symmetries in an interactive way sitting at a computer terminal.
The commands are chosen in such a way that a reader can understand what the effect of the procedures is.
The construction of VER(31) demonstrates in a nice way the power of solving these problems in an interactive way. Yet another advantage of doing computations interactively is that one might avoid expression-swell.
This section contains a complete terminal session of the computation of infinitesimal symmetries of Burgers' equation

$$u_t = uu_x + u_{xx} \tag{2.4.1}$$

The ideal of differential forms is defined on

$$\mathbb{R}^7 = \{(x(1),\ldots,x(7))\} = \{(x,t,u,u_x,u_t,u_{xt},u_{tt})\} \tag{2.4.2}$$

and is generated by

$$\begin{aligned} \alpha_1 &= du - u_x dx - u_t\, dt \\ \alpha_2 &= du_x - (u_t - uu_x)dx - u_{xt}dt \\ \alpha_3 &= du_t - u_{xt}\, dx - u_{tt}dt \end{aligned} \tag{2.4.3}$$

and their exterior derivatives.

Small characters refer to explicit REDUCE-commands, while capitals refer to results from the computer.
In this session we used a procedure P(*1), which prints equation VER(*1) and functions F(*), together with their arguments, present in that equation.

```
lisp depl!*:='
((liov (X 1) (X 2) (X 3) (X 4) (X 5) (X 6) (X 7))
 ((f 7) (X 1) (X 2) (X 3) (X 4) (X 5) (X 6) (X 7))
 ((f 6) (X 1) (X 2) (X 3) (X 4) (X 5) (X 6) (X 7))
 ((f 5) (X 1) (X 2) (X 3) (X 4) (X 5) (X 6) (X 7))
 ((f 4) (X 1) (X 2) (X 3) (X 4) (X 5) (X 6) (X 7))
 ((f 3) (X 1) (X 2) (X 3) (X 4) (X 5))
 ((f 2) (X 1) (X 2) (X 3) (X 4) (X 5))
 ((f 1) (X 1) (X 2) (X 3) (X 4) (X 5)))$

operator x,f,ver$

fload diffor$

GIVE VALUES TO D!@DIF AND VNAT(I) I=3,...,D! DIF (VNAT 1=X, VNAT
2= T)
in "opcoef.red","tools.red","ip.red","bevat.red","infsym.red"$
GIVE VALUE TO D!@DIF , TOTAL !
EXECUTE: ARRAY A!@IP(MAXIMAL DIMENSION);
d!@dif:=7$
array a! ip d!@dif$
for i:=1:d!@dif do vnat i:=x i$
alfa(1):=uit(3)-x(4)*uit(1)-x(5)*uit(2)$
alfa(2):=uit(4)-(x 5-x 3*x 4)*uit(1)-x(6)*uit(2)$
alfa(3):=uit(5)-x(6)*uit(1)-x(7)*uit(2)$
infsym(1,3,3);
THERE EXIST 2 EQUATIONS FOR ALFA(1)
THERE EXIST 4 EQUATIONS FOR ALFA(2)
THERE EXIST 4 EQUATIONS FOR ALFA(3)
10
off nat$

for i:=1:total do write ver i:=ver i$

VER(1) :=  - X(6)*X(5)*DF(F(2),X(5)) - X(6)*X(4)*DF(F(1),X(5
)) + X(6)*DF(F(3),X(5)) - X(5)**2*DF(F(2),X(4)) + X(5)*X(4)*
X(3)*DF(F(2),X(4)) - X(5)*X(4)*DF(F(2),X(3)) - X(5)*X(4)*DF(
F(1),X(4)) + X(5)*DF(F(3),X(4)) - X(5)*DF(F(2),X(1)) + X(4)
**2*X(3)*DF(F(1),X(4)) - X(4)**2*DF(F(1),X(3)) - X(4)*X(3)*
DF(F(3),X(4)) + X(4)*DF(F(3),X(3)) - X(4)*DF(F(1),X(1)) + DF
(F(3),X(1)) - F(4)$

VER(2) :=  - X(7)*X(5)*DF(F(2),X(5)) - X(7)*X(4)*DF(F(1),X(5
)) + X(7)*DF(F(3),X(5)) - X(6)*X(5)*DF(F(2),X(4)) - X(6)*X(4
)*DF(F(1),X(4)) + X(6)*DF(F(3),X(4)) - X(5)**2*DF(F(2),X(3))
 - X(5)*X(4)*DF(F(1),X(3)) + X(5)*DF(F(3),X(3)) - X(5)*DF(F(
2),X(2)) - X(4)*DF(F(1),X(2)) + DF(F(3),X(2)) - F(5)$

VER(3) :=  - X(6)**2*DF(F(2),X(5)) - X(6)*X(5)*DF(F(2),X(4))
 - X(6)*X(5)*DF(F(1),X(5)) + X(6)*X(4)*X(3)*DF(F(2),X(4)) +
X(6)*X(4)*X(3)*DF(F(1),X(5)) - X(6)*X(4)*DF(F(2),X(3)) + X(6
)*DF(F(4),X(5)) - X(6)*DF(F(2),X(1)) - X(5)**2*DF(F(1),X(4))
 + 2*X(5)*X(4)*X(3)*DF(F(1),X(4)) - X(5)*X(4)*DF(F(1),X(3))
 + X(5)*DF(F(4),X(4)) - X(5)*DF(F(1),X(1)) - X(4)**2*X(3)**2
*DF(F(1),X(4)) + X(4)**2*X(3)*DF(F(1),X(3)) - X(4)*X(3)*DF(F
(4),X(4)) + X(4)*X(3)*DF(F(1),X(1)) + X(4)*DF(F(4),X(3)) + X
(4)*F(3) + X(3)*F(4) + DF(F(4),X(1)) - F(5)$
```

```
VER(4) :=  - X(7)*X(6)*DF(F(2),X(5)) - X(7)*X(5)*DF(F(1),X(5
)) + X(7)*X(4)*X(3)*DF(F(1),X(5)) + X(7)*DF(F(4),X(5)) - X(6
)**2*DF(F(2),X(4)) - X(6)*X(5)*DF(F(2),X(3)) - X(6)*X(5)*DF(
F(1),X(4)) + X(6)*X(4)*X(3)*DF(F(1),X(4)) + X(6)*DF(F(4),X(4
)) - X(6)*DF(F(2),X(2)) - X(5)**2*DF(F(1),X(3)) + X(5)*X(4)*
X(3)*DF(F(1),X(3)) + X(5)*DF(F(4),X(3)) - X(5)*DF(F(1),X(2))
 + X(4)*X(3)*DF(F(1),X(2)) + DF(F(4),X(2)) - F(6)$

VER(5) := DF(F(4),X(6))$

VER(6) := DF(F(4),X(7))$

VER(7) :=  - X(7)*X(6)*DF(F(2),X(5)) - X(7)*X(5)*DF(F(2),X(4
)) + X(7)*X(4)*X(3)*DF(F(2),X(4)) - X(7)*X(4)*DF(F(2),X(3))
 - X(7)*DF(F(2),X(1)) - X(6)**2*DF(F(1),X(5)) - X(6)*X(5)*DF
(F(1),X(4)) + X(6)*X(4)*X(3)*DF(F(1),X(4)) - X(6)*X(4)*DF(F(
1),X(3)) + X(6)*DF(F(5),X(5)) - X(6)*DF(F(1),X(1)) + X(5)*DF
(F(5),X(4)) - X(4)*X(3)*DF(F(5),X(4)) + X(4)*DF(F(5),X(3))
 + DF(F(5),X(1)) - F(6)$

VER(8) :=  - X(7)**2*DF(F(2),X(5)) - X(7)*X(6)*DF(F(2),X(4))
 - X(7)*X(6)*DF(F(1),X(5)) - X(7)*X(5)*DF(F(2),X(3)) + X(7)*
DF(F(5),X(5)) - X(7)*DF(F(2),X(2)) - X(6)**2*DF(F(1),X(4))
 - X(6)*X(5)*DF(F(1),X(3)) + X(6)*DF(F(5),X(4)) - X(6)*DF(F(
1),X(2)) + X(5)*DF(F(5),X(3)) + DF(F(5),X(2)) - F(7)$

VER(9) := DF(F(5),X(6))$

VER(10) := DF(F(5),X(7))$

in difsol$

EXECUTE:ARRAY FIPKA1(MAX DIM)$$

array fipkal 7$

fpktel:=7$

fines 5$  % Comment: we first handle the equations which are easy to solve
HOMOGENEOUS INTEGRATION OF :(DF (F 4) (X 6))

fines 6$
HOMOGENEOUS INTEGRATION OF :(DF (F 8) (X 7))

f 4;

F(9)$
lisp car depl!*;

((F 9) (X 1) (X 2) (X 3) (X 4) (X 5))

fines 9$
HOMOGENEOUS INTEGRATION OF :(DF (F 5) (X 6))

fines 10$
HOMOGENEOUS INTEGRATION OF :(DF (F 10) (X 7))

fines 7$  % Comment: VER(7), VER(8) are solved first to avoid expression swell
```

```
VER(7) IS SOLVED FOR : (F 6)

fines 8$
VER(8) IS SOLVED FOR : (F 7)

fines 1$  % Comment: we now consider the remaining equations
VER(1) BREAKS INTO VER(11),...,VER(12) BY :
((X 6) (X 7))
p 12$  % Comment: it will turn out that VER(11) is not easy to handle at this
       %          moment

VER(12) :=  - X(5)**2*DF(F(2),X(4)) + X(5)*X(4)*X(3)*DF(F(2)
,X(4)) - X(5)*X(4)*DF(F(2),X(3)) - X(5)*X(4)*DF(F(1),X(4))
 + X(5)*DF(F(3),X(4)) - X(5)*DF(F(2),X(1)) + X(4)**2*X(3)*DF
(F(1),X(4)) - X(4)**2*DF(F(1),X(3)) - X(4)*X(3)*DF(F(3),X(4)
) + X(4)*DF(F(3),X(3)) - X(4)*DF(F(1),X(1)) + DF(F(3),X(1))
 - F(9)$
((F 2) (X 1) (X 2) (X 3) (X 4) (X 5))
((F 1) (X 1) (X 2) (X 3) (X 4) (X 5))
((F 3) (X 1) (X 2) (X 3) (X 4) (X 5))
((F 9) (X 1) (X 2) (X 3) (X 4) (X 5))

fines 12$
VER(12) IS SOLVED FOR : (F 9)

fines 2$
VER(2) BREAKS INTO VER(13),...,VER(15) BY :
((X 6) (X 7))

fines 15$
VER(15) IS SOLVED FOR : (F 11)

fines 3$
VER(3) BREAKS INTO VER(16),...,VER(18) BY :
((X 6) (X 7))

fines 4$
VER(4) BREAKS INTO VER(19),...,VER(22) BY :
((X 6) (X 7))

for i:=11:total do if ver i=0 then 0 else write ver i:=ver i$

VER(11) :=  - X(5)*DF(F(2),X(5)) - X(4)*DF(F(1),X(5)) + DF(F
(3),X(5))$

VER(13) :=  - X(5)*DF(F(2),X(4)) - X(4)*DF(F(1),X(4)) + DF(F
(3),X(4))$

VER(14) :=  - X(5)*DF(F(2),X(5)) - X(4)*DF(F(1),X(5)) + DF(F
(3),X(5))$

VER(16) :=  - DF(F(2),X(5))$

VER(17) :=  - X(5)**2*DF(F(2),X(5),X(4)) + X(5)*X(4)*X(3)*DF
(F(2),X(5),X(4)) - X(5)*X(4)*DF(F(2),X(5),X(3)) - X(5)*X(4)*
DF(F(1),X(5),X(4)) + X(5)*DF(F(3),X(5),X(4)) - X(5)*DF(F(2),
X(5),X(1)) - 3*X(5)*DF(F(2),X(4)) - X(5)*DF(F(1),X(5)) + X(4
)**2*X(3)*DF(F(1),X(5),X(4)) - X(4)**2*DF(F(1),X(5),X(3)) -
```

```
X(4)*X(3)*DF(F(3),X(5),X(4)) + 2*X(4)*X(3)*DF(F(2),X(4)) + X
(4)*X(3)*DF(F(1),X(5)) + X(4)*DF(F(3),X(5),X(3)) - 2*X(4)*DF
(F(2),X(3)) - X(4)*DF(F(1),X(5),X(1)) - X(4)*DF(F(1),X(4))
 + DF(F(3),X(5),X(1)) + DF(F(3),X(4)) - 2*DF(F(2),X(1))$


VER(18) :=  - X(5)**3*DF(F(2),X(4),2) + 2*X(5)**2*X(4)*X(3)*
DF(F(2),X(4),2) - 2*X(5)**2*X(4)*DF(F(2),X(4),X(3)) - X(5)**
2*X(4)*DF(F(1),X(4),2) + X(5)**2*DF(F(3),X(4),2) - 2*X(5)**2
*DF(F(2),X(4),X(1)) - 2*X(5)**2*DF(F(1),X(4)) - X(5)*X(4)**2
*X(3)**2*DF(F(2),X(4),2) + 2*X(5)*X(4)**2*X(3)*DF(F(2),X(4),
X(3)) + 2*X(5)*X(4)**2*X(3)*DF(F(1),X(4),2) + X(5)*X(4)**2*
DF(F(2),X(4)) - X(5)*X(4)**2*DF(F(2),X(3),2) - 2*X(5)*X(4)**
2*DF(F(1),X(4),X(3)) - 2*X(5)*X(4)*X(3)*DF(F(3),X(4),2) + 2*
X(5)*X(4)*X(3)*DF(F(2),X(4),X(1)) + 4*X(5)*X(4)*X(3)*DF(F(1)
,X(4)) + 2*X(5)*X(4)*DF(F(3),X(4),X(3)) - 2*X(5)*X(4)*DF(F(2
),X(3),X(1)) - 2*X(5)*X(4)*DF(F(1),X(4),X(1)) - 2*X(5)*X(4)*
DF(F(1),X(3)) - X(5)*X(3)*DF(F(2),X(1)) + 2*X(5)*DF(F(3),X(4
),X(1)) + X(5)*DF(F(2),X(2)) - X(5)*DF(F(2),X(1),2) - 2*X(5)
*DF(F(1),X(1)) - X(4)**3*X(3)**2*DF(F(1),X(4),2) + 2*X(4)**3
*X(3)*DF(F(1),X(4),X(3)) + X(4)**3*DF(F(1),X(4)) - X(4)**3*
DF(F(1),X(3),2) + X(4)**2*X(3)**2*DF(F(3),X(4),2) - 2*X(4)**
2*X(3)**2*DF(F(1),X(4)) - 2*X(4)**2*X(3)*DF(F(3),X(4),X(3))
 + 2*X(4)**2*X(3)*DF(F(1),X(4),X(1)) + 2*X(4)**2*X(3)*DF(F(1
),X(3)) - X(4)**2*DF(F(3),X(4)) + X(4)**2*DF(F(3),X(3),2) -
2*X(4)**2*DF(F(1),X(3),X(1)) - 2*X(4)*X(3)*DF(F(3),X(4),X(1)
) + X(4)*X(3)*DF(F(1),X(1)) + 2*X(4)*DF(F(3),X(3),X(1)) + X(
4)*DF(F(1),X(2)) - X(4)*DF(F(1),X(1),2) + X(4)*F(3) + X(3)*
DF(F(3),X(1)) - DF(F(3),X(2)) + DF(F(3),X(1),2)$

VER(19) :=  - DF(F(2),X(4)) + DF(F(1),X(5))$

VER(20) := X(5)**2*DF(F(2),X(5),X(3)) - X(5)**2*DF(F(2),X(4)
,2) + X(5)*X(4)*X(3)*DF(F(2),X(4),2) - X(5)*X(4)*DF(F(2),X(4
),X(3)) + X(5)*X(4)*DF(F(1),X(5),X(3)) - X(5)*X(4)*DF(F(1),X
(4),2) + X(5)*X(3)*DF(F(2),X(4)) - X(5)*DF(F(3),X(5),X(3))
 + X(5)*DF(F(3),X(4),2) + X(5)*DF(F(2),X(5),X(2)) - X(5)*DF(
F(2),X(4),X(1)) - X(5)*DF(F(1),X(4)) + X(4)**2*X(3)*DF(F(1),
X(4),2) - X(4)**2*DF(F(1),X(4),X(3)) - X(4)*X(3)*DF(F(3),X(4
),2) + 2*X(4)*X(3)*DF(F(1),X(4)) + X(4)*DF(F(3),X(4),X(3))
 + X(4)*DF(F(1),X(5),X(2)) - X(4)*DF(F(1),X(4),X(1)) - X(3)*
DF(F(3),X(4)) - DF(F(3),X(5),X(2)) + DF(F(3),X(4),X(1))$

VER(21) :=  - X(5)**2*DF(F(2),X(5),X(4)) + X(5)*X(4)*X(3)*DF
(F(2),X(5),X(4)) - X(5)*X(4)*DF(F(2),X(5),X(3)) - X(5)*X(4)*
DF(F(1),X(5),X(4)) + X(5)*DF(F(3),X(5),X(4)) - X(5)*DF(F(2),
X(5),X(1)) - X(5)*DF(F(2),X(4)) - X(5)*DF(F(1),X(5)) + X(4)
**2*X(3)*DF(F(1),X(5),X(4)) - X(4)**2*DF(F(1),X(5),X(3)) - X
(4)*X(3)*DF(F(3),X(5),X(4)) + X(4)*X(3)*DF(F(1),X(5)) + X(4)
*DF(F(3),X(5),X(3)) - X(4)*DF(F(1),X(5),X(1)) - X(4)*DF(F(1)
,X(4)) + DF(F(3),X(5),X(1)) + DF(F(3),X(4))$

VER(22) := X(5)*X(4)*(X(5)*DF(F(2),X(4)) + X(4)*DF(F(1),X(4)
) - DF(F(3),X(4)))$

fines 16$
HOMOGENEOUS INTEGRATION OF :(DF (F 2) (X 5))
```

```
p 19$

VER(19) :=  - DF(F(12),X(4)) + DF(F(1),X(5))$
((F 12) (X 1) (X 2) (X 3) (X 4))
((F 1) (X 1) (X 2) (X 3) (X 4) (X 5))

fines 19$
INHOMOGENEOUS INTEGRATION OF :(DF (F 1) (X 5))

p 11$

VER(11) :=  - X(4)*DF(F(12),X(4)) + DF(F(3),X(5))$
((F 12) (X 1) (X 2) (X 3) (X 4))
((F 3) (X 1) (X 2) (X 3) (X 4) (X 5))

fines 11$
INHOMOGENEOUS INTEGRATION OF :(DF (F 3) (X 5))

fines 13$
MORE THAN ONE MAXIMAL DF(F(*),*)

fines 14$
VER(14) IS ZERO

fines 17$
VER(17) BREAKS INTO VER(23),...,VER(24) BY :
((X 5) (X 6) (X 7))

fines 18$
VER(18) BREAKS INTO VER(25),...,VER(28) BY :
((X 5) (X 6) (X 7))

fines 20$
VER(20) BREAKS INTO VER(29),...,VER(30) BY :
((X 5) (X 6) (X 7))

fines 21$
MORE THAN ONE MAXIMAL DF(F(*),*)

fines 22$
MORE THAN ONE MAXIMAL DF(F(*),*)

for i:=13:total do if ver i=0 then 0 else  write ver i:=ver i$

VER(13) :=  - X(4)*DF(F(13),X(4)) + DF(F(14),X(4))$

VER(21) :=  - X(4)*DF(F(13),X(4)) + DF(F(14),X(4))$

VER(22) := X(5)*X(4)*(X(4)*DF(F(13),X(4)) - DF(F(14),X(4)))$

VER(23) :=  - 2*DF(F(12),X(4))$

VER(24) := 2*X(4)*X(3)*DF(F(12),X(4)) - X(4)*DF(F(13),X(4))
 - 2*X(4)*DF(F(12),X(3)) + DF(F(14),X(4)) - 2*DF(F(12),X(1))
$

VER(25) :=  - DF(F(12),X(4),2)$
```

```
VER(26) := 2*X(4)*X(3)*DF(F(12),X(4),2) - X(4)*DF(F(13),X(4)
,2) - 2*X(4)*DF(F(12),X(4),X(3)) + DF(F(14),X(4),2) - 2*DF(F
(13),X(4)) - 2*DF(F(12),X(4),X(1))$

VER(27) :=  - X(4)**2*X(3)**2*DF(F(12),X(4),2) + 2*X(4)**2*X
(3)*DF(F(13),X(4),2) + 2*X(4)**2*X(3)*DF(F(12),X(4),X(3)) -
2*X(4)**2*DF(F(13),X(4),X(3)) + X(4)**2*DF(F(12),X(4)) - X(4
)**2*DF(F(12),X(3),2) - 2*X(4)*X(3)*DF(F(14),X(4),2) + 4*X(4
)*X(3)*DF(F(13),X(4)) + 2*X(4)*X(3)*DF(F(12),X(4),X(1)) + 2*
X(4)*DF(F(14),X(4),X(3)) - 2*X(4)*DF(F(13),X(4),X(1)) - 2*X(
4)*DF(F(13),X(3)) - 2*X(4)*DF(F(12),X(3),X(1)) - X(3)*DF(F(
12),X(1)) + 2*DF(F(14),X(4),X(1)) - 2*DF(F(13),X(1)) + DF(F(
12),X(2)) - DF(F(12),X(1),2)$

VER(28) :=  - X(4)**3*X(3)**2*DF(F(13),X(4),2) + 2*X(4)**3*X
(3)*DF(F(13),X(4),X(3)) + X(4)**3*DF(F(13),X(4)) - X(4)**3*
DF(F(13),X(3),2) + X(4)**2*X(3)**2*DF(F(14),X(4),2) - 2*X(4)
**2*X(3)**2*DF(F(13),X(4)) - 2*X(4)**2*X(3)*DF(F(14),X(4),X(
3)) + 2*X(4)**2*X(3)*DF(F(13),X(4),X(1)) + 2*X(4)**2*X(3)*DF
(F(13),X(3)) - X(4)**2*DF(F(14),X(4)) + X(4)**2*DF(F(14),X(3
),2) - 2*X(4)**2*DF(F(13),X(3),X(1)) - 2*X(4)*X(3)*DF(F(14),
X(4),X(1)) + X(4)*X(3)*DF(F(13),X(1)) + 2*X(4)*DF(F(14),X(3)
,X(1)) + X(4)*DF(F(13),X(2)) - X(4)*DF(F(13),X(1),2) + X(4)*
F(14) + X(3)*DF(F(14),X(1)) - DF(F(14),X(2)) + DF(F(14),X(1)
,2)$

VER(29) :=  - X(4)*DF(F(13),X(4),2) + DF(F(14),X(4),2) - DF(
F(13),X(4))$

VER(30) := X(4)**2*X(3)*DF(F(13),X(4),2) - X(4)**2*DF(F(13),
X(4),X(3)) - X(4)*X(3)*DF(F(14),X(4),2) + 2*X(4)*X(3)*DF(F(
13),X(4)) + X(4)*DF(F(14),X(4),X(3)) - X(4)*DF(F(13),X(4),X(
1)) - X(3)*DF(F(14),X(4)) + DF(F(14),X(4),X(1))$

fines 23$
HOMOGENEOUS INTEGRATION OF :(DF (F 12) (X 4))

for i:=13:total do fines i$
MORE THAN ONE MAXIMAL DF(F(*),*)
VER(14) IS ZERO
VER(15) IS ZERO
VER(16) IS ZERO
VER(17) IS ZERO
VER(18) IS ZERO
VER(19) IS ZERO
VER(20) IS ZERO
MORE THAN ONE MAXIMAL DF(F(*),*)
MORE THAN ONE MAXIMAL DF(F(*),*)
VER(23) IS ZERO
MORE THAN ONE MAXIMAL DF(F(*),*)
VER(25) IS ZERO
MORE THAN ONE MAXIMAL DF(F(*),*)
MORE THAN ONE MAXIMAL DF(F(*),*)
MORE THAN ONE MAXIMAL DF(F(*),*)
MORE THAN ONE MAXIMAL DF(F(*),*)
MORE THAN ONE MAXIMAL DF(F(*),*)
% Comment: this fact demonstrates that the action of fines on
%          VER(13),...VER(30)  does not lead to new results!!!
```

```
p 26$

VER(26) :=  - X(4)*DF(F(13),X(4),2) + DF(F(14),X(4),2) - 2*
DF(F(13),X(4))$
((F 13) (X 1) (X 2) (X 3) (X 4))
((F 14) (X 1) (X 2) (X 3) (X 4))
p 29$

VER(29) :=  - X(4)*DF(F(13),X(4),2) + DF(F(14),X(4),2) - DF(
F(13),X(4))$
((F 13) (X 1) (X 2) (X 3) (X 4))
((F 14) (X 1) (X 2) (X 3) (X 4))

ver 31:=ver 26-ver 29;

VER(31) :=  - DF(F(13),X(4))$

total:=31$

fines 31$
HOMOGENEOUS INTEGRATION OF :(DF (F 13) (X 4))

fines 13$
HOMOGENEOUS INTEGRATION OF :(DF (F 14) (X 4))

fines 21$
VER(21) IS ZERO

fines 22$
VER(22) IS ZERO

p 24$

VER(24) :=  - 2*(X(4)*DF(F(15),X(3)) + DF(F(15),X(1)))$
((F 15) (X 1) (X 2) (X 3))

fines 24$
VER(24) BREAKS INTO VER(32),...,VER(33) BY :
((X 4) (X 5) (X 6) (X 7))

p 32$

VER(32) :=  - 2*DF(F(15),X(3))$
((F 15) (X 1) (X 2) (X 3))

fines 32$
HOMOGENEOUS INTEGRATION OF :(DF (F 15) (X 3))

p 33$

VER(33) :=  - 2*DF(F(18),X(1))$
((F 18) (X 1) (X 2))

fines 33$
HOMOGENEOUS INTEGRATION OF :(DF (F 18) (X 1))

fines 26$
VER(26) IS ZERO
```

```
fines 27$
VER(27) BREAKS INTO VER(34),...,VER(35) BY :
((X 4) (X 5) (X 6) (X 7))

p 34$

VER(34) :=  - 2*DF(F(16),X(3))$
((F 16) (X 1) (X 2) (X 3))

fines 34$
HOMOGENEOUS INTEGRATION OF :(DF (F 16) (X 3))

p 35$

VER(35) :=  - 2*DF(F(20),X(1)) + DF(F(19),X(2))$
((F 20) (X 1) (X 2))
((F 19) (X 2))

fines 35$
INHOMOGENEOUS INTEGRATION OF :(DF (F 20) (X 1))

fines 28$
VER(28) BREAKS INTO VER(36),...,VER(38) BY :
((X 4) (X 5) (X 6) (X 7))

p 36$

VER(36) := 2*DF(F(17),X(3),2)$
((F 17) (X 1) (X 2) (X 3))

fines 36$
HOMOGENEOUS INTEGRATION OF :(DF (F 17) (X 3) 2)

p 37$

VER(37) := X(3)*DF(F(19),X(2)) + 2*X(3)*F(22) + X(1)*DF(F(19
),X(2),2) + 4*DF(F(22),X(1)) + 2*DF(F(21),X(2)) + 2*F(23)$
((F 19) (X 2))
((F 22) (X 1) (X 2))
((F 21) (X 2))
((F 23) (X 1) (X 2))

fines 37$
VER(37) BREAKS INTO VER(39),...,VER(40) BY :
((X 3) (X 4) (X 5) (X 6) (X 7))

p 39$

VER(39) := DF(F(19),X(2)) + 2*F(22)$
((F 19) (X 2))
((F 22) (X 1) (X 2))

fines 39$
VER(39) IS SOLVED FOR : (F 22)

p 40$
```

```
VER(40) := X(1)*DF(F(19),X(2),2) + 2*DF(F(21),X(2)) + 2*F(23
)$
((F 19) (X 2))
((F 21) (X 2))
((F 23) (X 1) (X 2))


fines 40$
VER(40) IS SOLVED FOR : (F 23)

p 38$

VER(38) := X(1)*DF(F(19),X(2),3) + 2*DF(F(21),X(2),2)$
((F 19) (X 2))
((F 21) (X 2))

fines 38$
VER(38) BREAKS INTO VER(41),...,VER(42) BY :
((X 1) (X 3) (X 4) (X 5) (X 6) (X 7))

fines 41$
HOMOGENEOUS INTEGRATION OF :(DF (F 19) (X 2) 3)

fines 42$
HOMOGENEOUS INTEGRATION OF :(DF (F 21) (X 2) 2)

for i:=1:total do if ver i=0 then 0 else write ver i:=ver i$
         % Comment: all equations are zero, so the system has been solved

lisp vfpktrl:='(x t u)$
vfgen 3$
(F 28)
(F 27)
(F 26)
(F 25)
(F 24)
CREATION OF A TOTAL NUMBER OF 5  VECTOR FIELDS

in print$

on nat$

for i:=1:5 do write vf i:=vf i$


VF(1) := (D *X + 2*D *T - D *U)/2
           X         T      U

                            2
VF(2) := D *X*T + D *T  - D *(X + U*T)
          X        T       U
VF(3) := D
          T
VF(4) := D *T - D
          X      U
VF(5) := D
          X
```

Chapter 3

INFINITESIMAL SYMMETRIES OF PARTIAL DIFFERENTIAL EQUATIONS AND SOME OF THEIR APPLICATIONS

3.0 Introduction

In this chapter some applications of the software described in chapter 2 are given.

In section 1 the infinitesimal symmetries of vacuum Maxwell equations are computed. Moreover the problem of determining infinitesimal symmetries of vacuum Maxwell equations including potentials is solved. This problem was proposed by Estabrook at the Scheveningen conference on Geometrical Approaches to Differential Equations (1979) [9].
In section 2 the Lie algebra of infinitesimal symmetries of nonlinear diffusion equation is given for all distinct values of the parameters occurring in this equation.
In section 3 the infinitesimal symmetries of (3 + 1)-nonlinear Schrödinger equation are described.
In section 4 we study the Lie algebra of infinitesimal symmetries of 4 cases of nonlinear Dirac equations. In addition conserved currents associated to new symmetries are constructed.
Finally, in section 5 we derive the Lie algebra of infinitesimal symmetries of self-dual SU(2) Yang-Mills Equations and of the corresponding static gauge field.
By taking combinations of rotations and gauge transformations we obtain the Belavin-Polyakov-Schwartz-Tyupkin instanton solution and the Prasad-Sommerfield monopole solution for the static gauge field.

3.1 Infinitesimal symmetries of vacuum Maxwell equations

In [12] Harrison & Estabrook derived the Lie algebra of infinitesimal symmetries of vacuum Maxwell equations,

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \nabla \times \underline{B} = \frac{\partial \underline{E}}{\partial t}$$

$$\nabla \cdot \underline{B} = 0 \qquad \nabla \cdot \underline{E} = 0 \tag{3.1.1}$$

where $\underline{E} = (E_1, E_2, E_3)$, $\underline{B} = (B_1, B_2, B_3)$ are the electric and magnetic field, $\nabla = (\partial_x, \partial_y, \partial_z)$, $\underline{x} = (x,y,z)$.

Their construction started at an ideal I of differential forms in 10-dimensional space $\mathbb{R}^{10} = \{(x(1),\dots,x(10)\} = \{(t,\underline{x},\underline{E},\underline{B})\}$ generated by two 3-forms

$$\begin{aligned} \alpha_2 &= dx(5)dx(2)dx(1) + dx(6)dx(3)dx(1) + dx(7)dx(4)dx(1) \\ &\quad + dx(8)dx(3)dx(4) + dx(9)dx(4)dx(2) + dx(10)dx(2)dx(3) \\ \alpha_3 &= dx(8)dx(2)dx(1) + dx(9)dx(3)dx(1) + dx(10)dx(4)dx(1) \\ &\quad - dx(5)dx(3)dx(4) - dx(6)dx(4)dx(2) - dx(7)dx(2)dx(3). \end{aligned} \tag{3.1.2}$$

In the computation of the infinitesimal symmetries they introduced complex vector calculus in order to diminish the algebraic labour.

Let the vector field V be defined by

```
V:= FOR I:= 1 : 10 SUM F(I)*D↑I                                (3.1.3)
```

The condition on V to be an infinitesimal symmetry is that the following two expressions(c.f. 1.1.26) vanish

```
LIEDF(V,ALFA(2)) + A2 * ALFA(2) + A3 * ALFA(3)
                                                               (3.1.4)
LIEDF(V,ALFA(3)) + B2 * ALFA(2) + B3 * ALFA(3)
```

We now equate the coefficients of the basis 3-forms in the two expressions (3.1.4) to zero, using the procedure OPCOEFF [10], which lead to an over-determined system of 176 partial differential equations for the components of the vector field V, F(I) (I=1,...,10).

From this overdetermined system we obtain the following results (c.f.[17])

1 : F(1),...,F(4) are polynomials of maximum degree 3 in x(1),...,x(4), and do not depend on x(5),...,x(10).

2 : F(5),...,F(10) can be obtained from F(1),...,F(4) by quadrature introducing 6 functions F(5,VAR),...,F(10,VAR) which depend on x(1),...,x(4) and which satisfy vacuum Maxwell equations (3.1.1) due to the linearity of the problem [12]. The functions F(5,VAR),..,F(10,VAR) lead to the infinitesimal symmetry

$$F(5,VAR)\,\partial_{x(5)} + \ldots + F(10,VAR)\,\partial_{x(10)} \qquad (3.1.5)$$

Substitution of the results 1,2 into the set of equations and equating coefficients of like powers in x(1),...,x(4) to zero leads to a 17-dimensional Lie algebra of infinitesimal symmetries, and moreover the so called continuous part (3.1.5) due to the linearity i.e.

$$
\begin{aligned}
VF(1) &:= D_T \\
VF(2) &:= D_X \\
VF(3) &:= D_Y \\
VF(4) &:= D_Z \\
VF(5) &:= D_T*X + D_X*T + D_{E2}*B3 - D_{E3}*B2 - D_{B2}*E3 + D_{B3}*E2 \\
VF(6) &:= D_T*Y + D_Y*T - D_{E1}*B3 + D_{E3}*B1 + D_{B1}*E3 - D_{B3}*E1 \\
VF(7) &:= D_T*Z + D_Z*T + D_{E1}*B2 - D_{E2}*B1 - D_{B1}*E2 + D_{B2}*E1 \\
VF(8) &:= D_X*Y - D_Y*X + D_{E1}*E2 - D_{E2}*E1 + D_{B1}*B2 - D_{B2}*B1 \\
VF(9) &:= D_X*Z - D_Z*X + D_{E1}*E3 - D_{E3}*E1 + D_{B1}*B3 - D_{B3}*B1 \qquad (3.1.6) \\
VF(10) &:= D_Y*Z - D_Z*Y + D_{E2}*E3 - D_{E3}*E2 + D_{B2}*B3 - D_{B3}*B2 \\
VF(11) &:= D_T*(X^2 + Y^2 + Z^2 + T^2) + 2*D_X*X*T + 2*D_Y*Y*T + 2*D_Z*Z*T \\
&\quad + 2*D_{E1}*(- 2*E1*T + B2*Z - B3*Y) + 2*D_{E2}*(- 2*E2*T \\
&\quad - B1*Z + B3*X) + 2*D_{E3}*(- 2*E3*T + B1*Y - B2*X) + \\
&\quad 2*D_{B1}*(- E2*Z + E3*Y - 2*B1*T) + 2*D_{B2}*(E1*Z - E3*X - \\
&\quad 2*B2*T) + 2*D_{B3}*(- E1*Y + E2*X - 2*B3*T)
\end{aligned}
$$

$$VF(12) := 2*D_{T}*X*T + D_{X}*(X^{2} - Y^{2} - Z^{2} + T^{2}) + 2*D_{Y}*X*Y + 2*D_{Z}*X*Z + 2*D_{E1}*(- 2*E1*X - E2*Y - E3*Z) + 2*D_{E2}*(E1*Y - 2*E2*X + B3*T) + 2*D_{E3}*(E1*Z - 2*E3*X - B2*T) + 2*D_{B1}*(- 2*B1*X - B2*Y - B3*Z) + 2*D_{B2}*(- E3*T + B1*Y - 2*B2*X) + 2*D_{B3}*(E2*T + B1*Z - 2*B3*X)$$

$$VF(13) := 2*D_{T}*Y*T + 2*D_{X}*X*Y + D_{Y}*(- X^{2} + Y^{2} - Z^{2} + T^{2}) + 2*D_{Z}*Y*Z + 2*D_{E1}*(- 2*E1*Y + E2*X - B3*T) + 2*D_{E2}*(- E1*X - 2*E2*Y - E3*Z) + 2*D_{E3}*(E2*Z - 2*E3*Y + B1*T) + 2*D_{B1}*(E3*T - 2*B1*Y + B2*X) + 2*D_{B2}*(- B1*X - 2*B2*Y - B3*Z) + 2*D_{B3}*(- E1*T + B2*Z - 2*B3*Y)$$

$$VF(14) := 2*D_{T}*Z*T + 2*D_{X}*X*Z + 2*D_{Y}*Y*Z + D_{Z}*(- X^{2} - Y^{2} + Z^{2} + T^{2}) + 2*D_{E1}*(- 2*E1*Z + E3*X + B2*T) + 2*D_{E2}*(- 2*E2*Z + E3*Y - B1*T) + 2*D_{E3}*(- E1*X - E2*Y - 2*E3*Z) + 2*D_{B1}*(- E2*T - 2*B1*Z + B3*X) + 2*D_{B2}*(E1*T - 2*B2*Z + B3*Y) + 2*D_{B3}*(- B1*X - B2*Y - 2*B3*Z)$$

$$VF(15) := - D_{E1}*E1 - D_{E2}*E2 - D_{E3}*E3 - D_{B1}*B1 - D_{B2}*B2 - D_{B3}*B3$$

$$VF(16) := - D_{E1}*B1 - D_{E2}*B2 - D_{E3}*B3 + D_{B1}*E1 + D_{B2}*E2 + D_{B3}*E3$$

$$VF(17) := D_{T}*T + D_{X}*X + D_{Y}*Y + D_{Z}*Z - 2*D_{E1}*E1 - 2*D_{E2}*E2 - 2*D_{E3}*E3 - 2*D_{B1}*B1 - 2*D_{B2}*B2 - 2*D_{B3}*B3$$

We now arrive at the computation of the infinitesimal symmetries of vacuum Maxwell equations, including potentials; i.e.,

$$\begin{array}{ll} \nabla\cdot\underline{B} = 0 & \nabla\cdot\underline{E} = 0 \\ \nabla\times\underline{E} = -\dfrac{\partial\underline{B}}{\partial t} & \nabla\times\underline{B} = \dfrac{\partial\underline{E}}{\partial t} \\ \underline{E} = \nabla\times\underline{A} & \dfrac{\partial\phi}{\partial t} - \nabla\underline{A} = 0 \\ \underline{B} = \dfrac{\partial\underline{A}}{\partial t} - \nabla\phi & \end{array} \qquad (3.1.7)$$

Let I' be the ideal of differential forms in 14-dimensional space

$\mathbb{R}^{14} = \{(x(1),\ldots,x(14))\} = \{(t,\underline{x},\underline{E},\underline{B},\phi,\underline{A})\}$ generated by [9]

$$\begin{aligned}
\alpha_1 &= d(x(11)dx(1) + x(12)dx(2) + x(13)dx(3) + x(14)dx(4)) \\
&\quad - x(5)dx(3)dx(4) - x(6)dx(4)dx(2) - x(7)dx(2)dx(3) \\
&\quad + x(8)dx(2)dx(1) + x(9)dx(3)dx(1) + x(10)dx(4)dx(1) \\
\alpha_2 &= dx(5)dx(2)dx(1) + dx(6)dx(3)dx(1) + dx(7)dx(4)dx(1) \\
&\quad + dx(8)dx(3)dx(4) + dx(9)dx(4)dx(2) + dx(10)dx(2)dx(3) \\
\alpha_3 &= dx(8)dx(2)dx(1) + dx(9)dx(3)dx(1) + dx(10)dx(4)dx(1) \\
&\quad - dx(5)dx(3)dx(4) - dx(6)dx(4)dx(2) - dx(7)dx(2)dx(3) \\
\alpha_4 &= dx(11)dx(2)dx(3)dx(4) + dx(12)dx(3)dx(4)dx(1) \\
&\quad + dx(13)dx(4)dx(2)dx(1) + dx(14)dx(2)dx(3)dx(1)
\end{aligned} \qquad (3.1.8)$$

Let S_4 be a 4-dimensional submanifold of $\mathbb{R}^{14}$ and moreover let

$$dx(1)dx(2)dx(3)dx(4) \neq 0$$

on S_4.
Then it follows that S_4 can be parametrized by $x(1),\ldots,x(4)$, and a straightforward calculation [17] shows that S_4 is an integral manifold of the ideal I' (3.1.8) i.e.,

$$I' \Big|_{S_4} = 0$$

if and only if in local coordinates $x(1),\ldots,x(4)$ $(x(5),\ldots,x(14)) = (\underline{E},\underline{B},\underline{\phi},\underline{A})$ satisfy (3.1.7).
The infinitesimal symmetries of I' have to satisfy the conditions

$$L_V\, \alpha_i \in I' \quad (i=1,\ldots,4). \qquad (3.1.9)$$

These conditions lead to 76, 136, 64, 455 partial differential equations for the components of the vector field V. The partial results obtained from the first set of equations were used in the following steps and so on.

Similar calculations as those carried out to obtain the result (3.1.6) lead to the following 12-dimensional Lie algebra of infinitesimal symmetries of the ideal I' i.e.,

$$VF(1) := D_T$$

$$VF(2) := D_X$$

$$VF(3) := D_Y$$

$$VF(4) := D_Z$$

$$VF(5) := D_T*X + D_X*T + D_{E2}*B3 - D_{E3}*B2 - D_{B2}*E3 + D_{B3}*E2 - D_{FI}*A1 - D_{A1}*FI$$

$$VF(6) := D_T*Y + D_Y*T - D_{E1}*B3 + D_{E3}*B1 + D_{B1}*E3 - D_{B3}*E1 - D_{FI}*A2 - D_{A2}*FI \qquad (3.1.10)$$

$$VF(7) := D_T*Z + D_Z*T + D_{E1}*B2 - D_{E2}*B1 - D_{B1}*E2 + D_{B2}*E1 - D_{FI}*A3 - D_{A3}*FI$$

$$VF(8) := D_X*Y - D_Y*X + D_{E1}*E2 - D_{E2}*E1 + D_{B1}*B2 - D_{B2}*B1 + D_{A1}*A2 - D_{A2}*A1$$

$$VF(9) := D_X*Z - D_Z*X + D_{E1}*E3 - D_{E3}*E1 + D_{B1}*B3 - D_{B3}*B1 + D_{A1}*A3 - D_{A3}*A1$$

$$VF(10) := D_Y*Z - D_Z*Y + D_{E2}*E3 - D_{E3}*E2 + D_{B2}*B3 - D_{B3}*B2 + D_{A2}*A3 - D_{A3}*A2$$

$$VF(11) := - D_{E1}*E1 - D_{E2}*E2 - D_{E3}*E3 - D_{B1}*B1 - D_{B2}*B2 - D_{B3}*B3 - D_{FI}*FI - D_{A1}*A1 - D_{A2}*A2 - D_{A3}*A3$$

$$VF(12) := D_T*T + D_X*X + D_Y*Y + D_Z*Z - 2*D_{E1}*E1 - 2*D_{E2}*E2 - 2*D_{E3}*E3 - 2*D_{B1}*B1 - 2*D_{B2}*B2 - 2*D_{B3}*B3 - D_{FI}*FI - D_{A1}*A1 - D_{A2}*A2 - D_{A3}*A3$$

and a continuous part generated by functions F(j,VAR) (j=5,...,14) dependent on x(1),...,x(4), which satisfy (3.1.7), due to the linearity.
In [17], where more details can be found we used a somewhat different notation.

3.2 Infinitesimal symmetries of nonlinear diffusion equation

The (3 + 1)-nonlinear diffusion equation is given by

$$\Delta(u^{p+1}) + ku^q = u_t \tag{3.2.1}$$

where $u = u(x,y,z,t)$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$; $p,k,q \in \mathbb{Q}(p \neq -1)$

We shall derive the Lie algebras of infinitesimal symmetries for all distinct values of p, k, q.

Let I be the differential ideal of differential forms in

$\mathbb{R}^{18} = \{(x(1),\ldots,x(18))\} = \{(x,y,z,t,u_x,\ldots,\hat{u}_{zz},\ldots,u_{tt})\}$,
(^ means deletion),

generated by

$$\begin{aligned}
\alpha_1 &= du - u_x\, dx - u_y\, dy - u_z\, dz - u_t\, dt\\
\alpha_2 &= du_x - u_{xx}\, dx - u_{xy}\, dy - u_{xz}\, dz - u_{xt}\, dt\\
\alpha_3 &= du_y - u_{xy}\, dx - u_{yy}\, dy - u_{yz}\, dz - u_{yt}\, dt\\
\alpha_4 &= du_z - u_{xz}\, dx - u_{yz}\, dy - u_{zz}\, dz - u_{zt}\, dt\\
\alpha_5 &= du_t - u_{xt}\, dx - u_{yt}\, dy - u_{zt}\, dz - u_{tt}\, dt
\end{aligned} \tag{3.2.2}$$

where u_{zz} is obtained by solving (3.2.1) i.e.,

$$u_{zz} = -u_{xx} - u_{yy} - pu^{-1}(u_x^2+u_y^2+u_z^2) + (p+1)^{-1}u^{-p}(u_t-ku^q) \tag{3.2.2a}$$

In order to compute the Lie algebra of infinitesimal symmetries of the ideal I (3.2.2) we used the software described in chapter 2.
For more details of the computation we refer to [19].

In the various steps of the computation it becomes evident that we have to distinguish 9 combinations of values of the parameters p, k, q, giving rise to different Lie algebras.

First of all, we formulate two general results

1 Nonlinear diffusion equation (3.2.1) does not admit more general Lie contact symmetries than Lie point symmetries [1], [19].

2 For any value of p, k, q, equation (3.2.1) admits the following 7 infinitesimal symmetries

$$X_1 = \partial_x \; ; \; X_2 = \partial_y \; ; \; X_3 = \partial_z \; ; \; X_4 = \partial_t \; ;$$
$$X_5 = y\partial_x - x\partial_y \; ; \; X_6 = z\partial_x - x\partial_z \; ; \; X_7 = z\partial_y - y\partial_z . \tag{3.2.3}$$

We now summarize the final results, while the complete Lie algebras are given for the cases, which have to be distinguished:

case 1 : p = 0, k = 0

The complete Lie algebra of infinitesimal symmetries of

$$\Delta(u) = u_t \tag{3.2.4}$$

is spanned by $X_1,\ldots,X_7$ given in (3.2.3)

and

$$\begin{aligned}
X_8 &= u\partial_u \\
X_9 &= 2t\partial_x - xu\partial_u \\
X_{10} &= 2t\partial_y - yu\partial_u \\
X_{11} &= 2t\partial_z - zu\partial_u \\
X_{12} &= x\partial_x + y\partial_y + z\partial_z + 2t\partial_t \\
X_{13} &= xt\partial_x + yt\partial_y + zt\partial_z + t^2\partial_t + \tfrac{1}{4}\, u(-x^2-y^2-z^2-6t)\partial_u
\end{aligned} \tag{3.2.4a}$$

and the continous part $F(x,y,z,t)\partial_u$;

F satisfies the PDE : $\Delta(F) = F_t$.

case 2 : p = 0, k ≠ 0, q = 1

The complete Lie algebra of infinitesimal symmetries of

$$\Delta(u)+ku=u_t \tag{3.2.5}$$

is spanned by $X_1,\ldots,X_7$ given in (3.2.3)
and

$$\begin{aligned}
X_8 &= u\partial_u , \\
X_9 &= 2t\partial_x - xu\partial_u , \\
X_{10} &= 2t\partial_y - yu\partial_u \\
X_{11} &= 2t\partial_z - zu\partial_u \\
X_{12} &= x\partial_x + y\partial_y + z\partial_z + 2t\partial_t + 2kut\partial_u \\
X_{13} &= xt\partial_x + yt\partial_t + zt\partial_z + t^2\partial_t + \frac{1}{4}u(4kt^2-x^2-y^2-z^2-6t)\partial_u
\end{aligned} \quad (3.2.5a)$$

and the continuous part $F(x,y,z,t)\partial_u$;

F satisfies the P.D.E: $\Delta(F) + kF = F_t$

case 3 : $p = 0,\ k \neq 0,\ q \neq 1$

The complete Lie algebra of infinitesimal symmetries of

$$\Delta(u)+ku^q=u_t \quad (3.2.6)$$

is spanned by $X_1,\ldots,X_7$ given in (3.2.3)
and

$$X_8 = x\partial_x + y\partial_y + z\partial_z + 2t\partial_t - \frac{2}{q-1}u\partial_u \quad (3.2.6a)$$

case 4 : $p = -\frac{4}{5}$, $k = 0$

The complete Lie algebra of infinitesimal symmetries of

$$\Delta(u^{\frac{1}{5}}) = u_t \quad (3.2.7)$$

is spanned by $X_1,\ldots,X_7$ given in (3.2.3)
and

$$\begin{aligned}
X_8 &= 4t\partial_t + 5u\partial_u \\
X_9 &= 2x\partial_x + 2y\partial_y + 2z\partial_z - 5u\partial_u \\
X_{10} &= (+x^2-y^2-z^2)\partial_x + 2xy\partial_y + 2xz\partial_z - 5xu\partial_u \\
X_{11} &= 2xy\partial_x + (-x^2+y^2-z^2)\partial_y + 2yz\partial_z - 5yu\partial_u \\
X_{12} &= 2xz\partial_x + 2yz\partial_y + (-x^2-y^2+z^2)\partial_z - 5zu\partial_u
\end{aligned} \quad (3.2.7a)$$

case 5 : $p \neq -\frac{4}{5}$, $p \neq 0$, $k = 0$

The complete Lie algebra of infinitesimal symmetries of

$$\Delta(u^{p+1}) = u_t \tag{3.2.8}$$

is spanned by $X_1,\ldots,X_7$ given in (3.2.3)

and

$$\begin{aligned} X_8 &= -pt\partial_t + u\partial_u \\ X_9 &= px\partial_x + py\partial_y + pz\partial_z + 2u\partial_u \end{aligned} \tag{3.2.8a}$$

case 6 : $p = -\frac{4}{5}$, $k \neq 0$, $q = 1$

The complete Lie algebra of infinitesimal symmetries of

$$\Delta(u^{\frac{1}{5}}) + ku = u_t \tag{3.2.9}$$

is spanned by $X_1,\ldots,X_7$ given in (3.2.3)

and

$$\begin{aligned} X_8 &= e^{\frac{4kt}{5}}\partial_t + ku\, e^{\frac{4kt}{5}}\partial_u \\ X_9 &= 2x\partial_x + 2y\partial_y + 2z\partial_z - 5u\partial_u \\ X_{10} &= (+x^2-y^2-z^2)\partial_x + 2xy\partial_y + 3xz\partial_z - 5xu\partial_u \\ X_{11} &= 2xy\partial_x + (-x^2+y^2-z^2)\partial_y + 2yz\partial_z - 5yu\partial_u \\ X_{12} &= 2xz\partial_x + 2yz\partial_y + (-x^2-y^2+z^2)\partial_z - 5zu\partial_u \end{aligned} \tag{3.2.9a}$$

case 7 : $p \neq 0$, $p \neq -\frac{4}{5}$, $k \neq 0$, $q = 1$

The complete Lie algebra of infinitesimal symmetries of

$$\Delta(u^{p+1}) + ku = u_t \tag{3.2.10}$$

is spanned by $X_1,\ldots,X_7$ given in (3.2.3)

and

$$\begin{aligned} X_8 &= e^{-pkt}(\partial_t + ku\partial_u) \\ X_9 &= px\partial_x + py\partial_y + pz\partial_z + 2u\partial_u \end{aligned} \tag{3.2.10a}$$

case 8 : $p \neq 0$, $p \neq -\frac{4}{5}$, $q = p + 1$

The complete Lie algebra of infinitesimal symmetries of

$$\Delta(u^{p+1}) + ku^{p+1} = u_t \qquad (3.2.11)$$

is spanned by $X_1, \ldots, X_7$ given in (3.2.3)

and

$$X_8 = pt\partial_t - u\partial_u \qquad (3.2.11a)$$

case 9 : $p \neq 0$; $p \neq -\frac{4}{5}$, $q \neq 1$, $q \neq p + 1$

The complete Lie algebra of infinitesimal symmetries of

$$\Delta(u^{p+1}) + ku^q = u_t \qquad (3.2.12)$$

is spanned by $X_1, \ldots, X_7$ (3.2.3)

and

$$X_8 = (-p+q-1)(x\partial_x + y\partial_y + z\partial_z) + 2(q-1)t\partial_t - 2u\partial_u \qquad (3.2.12a)$$

The results in these 9 cases are a generalization of the results of Branson & Steeb [6] in case N = 3, N being the number of space variables.

3.3 Infinitesimal symmetries of (3 + 1)-nonlinear Schrödinger equation

(3 + 1)-nonlinear Schrödinger equation is given by

$$i\,\frac{\partial f}{\partial t} + \Delta f = -\varepsilon f^2 f^* \qquad (3.3.1)$$

where in (3.3.1) $f = u + iv$, $f^* = u - iv$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. (3.3.1a)

Rewriting (3.3.1) in terms of u and v results in

$$\begin{aligned} -v_t + \Delta u &= -\varepsilon u(u^2+v^2) \\ u_t + \Delta v &= -\varepsilon v(u^2+v^2). \end{aligned} \qquad (3.3.2)$$

A differential ideal I associated to (3.3.2) in 32-dimensional space

$$\mathbb{R}^{32}=\{(x(1),\dots,x(32))\} = \{(\underline{x},u,v,u_{\underline{x}},v_{\underline{x}},u_{\underline{xx}},\hat{u}_{zz},v_{\underline{xx}},\hat{v}_{zz})\} \qquad (3.3.3)$$

is generated by the 10 1-forms

$$\begin{aligned}
\alpha(1) &= du - u_x\,dx - u_y\,dy - u_z\,dz - u_t\,dt \\
\alpha(2) &= dv - v_x\,dx - v_y\,dy - v_z\,dz - v_t\,dt \\
\alpha(3) &= du_x - u_{xx}\,dx - u_{xy}\,dy - u_{xz}\,dz - u_{xt}\,dt \\
\alpha(4) &= du_y - u_{xy}\,dx - u_{yy}\,dy - u_{yz}\,dz - u_{yt}\,dt \\
\alpha(5) &= du_z - u_{xz}\,dx - u_{yz}\,dy - u_{zz}\,dz - u_{zt}\,dt \\
\alpha(6) &= du_t - u_{xt}\,dx - u_{yt}\,dy - u_{zt}\,dz - u_{tt}\,dt \\
\alpha(7) &= dv_x - v_{xx}\,dx - x_{xy}\,dy - v_{xz}\,dz - v_{xt}\,dt \\
\alpha(8) &= dv_y - v_{xy}\,dx - v_{yy}\,dy - v_{yz}\,dz - v_{yt}\,dt \\
\alpha(9) &= dv_z - v_{xz}\,dx - v_{yz}\,dy - v_{zz}\,dz - v_{zt}\,dt \\
\alpha(10) &= dv_t - v_{xt}\,dx - v_{yt}\,dy - v_{zt}\,dz - v_{tt}\,dt
\end{aligned} \qquad (3.3.4)$$

and substitution of u_{zz}, v_{zz} obtained from (3.3.2) into (3.3.4)
In (3.3.3) $\hat{u}_{zz}$, $\hat{v}_{zz}$ means deletion, due to the partial differential equation (3.3.2).

For the Lie algebra of infinitesimal symmetries of (3.3.2) we derive the following result

Case 1 ($\varepsilon=0$) Linear Schrödinger equation

The Lie algebra of infinitesimal symmetries of (3.3.2) is spanned by 14 generators. Moreover there is a continuous part, due to the linearity of (3.3.1 $\varepsilon=0$)

The generators are given by

$$
\begin{aligned}
VF(1) &:= D_X \\
VF(2) &:= D_Y \\
VF(3) &:= D_Z \\
VF(4) &:= D_T \\
VF(5) &:= D_X*Y - D_Y*X \\
VF(6) &:= - D_X*Z + D_Z*X \\
VF(7) &:= - D_Y*Z + D_Z*Y \\
VF(8) &:= 2*D_X*T - D_U*V*X + D_V*X*U \\
VF(9) &:= 2*D_Y*T - D_U*V*Y + D_V*Y*U \\
VF(10) &:= 2*D_Z*T - D_U*V*Z + D_V*Z*U \\
VF(11) &:= D_U*U + D_V*V \\
VF(12) &:= - D_U*V + D_V*U \\
VF(13) &:= D_X*X + D_Y*Y + D_Z*Z + 2*D_T*T \\
VF(14) &:= 4*D_X*X*T + 4*D_Y*Y*T + 4*D_Z*Z*T + 4*D_T*T^2 + \\
&\quad D_U*(- R^2*V - 6*U*T) + D_V*(R^2*U - 6*V*T)
\end{aligned}
\tag{3.3.5}
$$

where $R^2 = X^2 + Y^2 + Z^2$.

The result is in agreement with those obtained by Niederer [26] and Barut [2].

Case 2 ($\varepsilon \neq 0$) Nonlinear Schrödinger equation.
The Lie algebra of infinitesimal symmetries of nonlinear Schrödinger equation is spanned by 12 generators i.e.,

$$VF(1),\ldots,VF(10),\ VF(11) - \frac{1}{2}\ VF(13),\ VF(14). \qquad (3.3.6)$$

The result (3.3.6) is similar to the one obtained by Kumei [22], where as a side product in the search for Lie-Bäcklund transformations the results of the (1 + 1)-nonlinear Schrödinger equation are given.

3.4 Infinitesimal symmetries and conserved currents for nonlinear Dirac equations

The Lie algebras of infinitesimal symmetries of Dirac equations, Dirac equations with nonvanishing rest mass and nonlinear Dirac equations are established.
In addition conserved currents associated to new symmetries are computed.

We shall only give a small survey of the solution procedure, for more details of the derivation we refer to [16].

The first purpose is the construction of the infinitesimal symmetries of Dirac equations [4]

$$\sum_{k=1}^{3} \hbar \frac{\partial}{\partial x_k} (\gamma_k \psi) - i\hbar \frac{\partial}{\partial x_4} (\gamma_4 \psi) + m_0 c\psi + n_0 \psi(\bar{\psi}\psi) = 0 \qquad (3.4.1)$$

where in (3.4.1)

$x_4 = ct$, $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)^T$, (T means transposed),

$\bar{\psi} = (\overset{*}{\psi}_1, \overset{*}{\psi}_2, -\overset{*}{\psi}_3, -\overset{*}{\psi}_4)$;

and $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are 4×4 matrices defined as

$$\gamma_1 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}, \quad \gamma_2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$
$$\gamma_3 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix}, \quad \gamma_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \qquad (3.4.1a)$$

After the introduction of

$$\lambda = \frac{\hbar}{m_0 c}$$

we obtain

$$\lambda \sum_{k=1}^{3} \frac{\partial}{\partial x_k} (\gamma_k \psi) - \lambda\, i \frac{\partial}{\partial x_4} (\gamma_4 \psi) + \psi + \lambda^3 \varepsilon\, \psi(\bar{\psi}\psi) = 0. \tag{3.4.2}$$

In the derivation of the infinitesimal symmetries of (3.4.2) we have to distinguish the following cases

$$\begin{aligned}
&\underline{1} : \varepsilon = 0 \quad , \quad \lambda^{-1} = 0 : \text{Dirac equations with vanishing rest mass.}\\
&\underline{2} : \varepsilon = 0 \quad , \quad \lambda^{-1} \neq 0 : \text{Dirac equations with nonvanishing rest mass.}\\
&\underline{3} : \varepsilon \neq 0 \quad , \quad \lambda^{-1} \neq 0 : \text{nonlinear Dirac equations.}\\
&\underline{4} : \varepsilon \neq 0 \quad , \quad \lambda^{-1} = 0 : \text{nonlinear Dirac equations with vanishing rest mass.}
\end{aligned} \tag{3.4.3}$$

We now put $\psi_j = u^j + iv^j$ $(j=1,\dots,4)$ into (4.2) and obtain a system of 8 coupled partial differential equations.

$$\begin{aligned}
\lambda\, v_1^4 - \lambda\, u_2^4 + \lambda\, v_3^3 + \lambda\, v_4^1 + (1+\lambda^3 \varepsilon K)\, u^1 &= 0\\
\lambda\, v_1^3 + \lambda\, u_2^3 - \lambda\, v_3^4 + \lambda\, v_4^2 + (1+\lambda^3 \varepsilon K)\, u^2 &= 0\\
-\lambda\, v_1^2 + \lambda\, u_2^2 - \lambda\, v_3^1 - \lambda\, v_4^3 + (1+\lambda^3 \varepsilon K)\, u^3 &= 0\\
-\lambda\, v_1^1 - \lambda\, u_2^1 + \lambda\, v_3^2 - \lambda\, v_4^4 + (1+\lambda^3 \varepsilon K)\, u^4 &= 0\\
-\lambda\, u_1^4 - \lambda\, v_2^4 - \lambda\, u_3^3 - \lambda\, u_4^1 + (1+\lambda^3 \varepsilon K)\, v^1 &= 0\\
-\lambda\, u_1^3 + \lambda\, v_2^3 + \lambda\, u_3^4 - \lambda\, u_4^2 + (1+\lambda^3 \varepsilon K)\, v^2 &= 0\\
\lambda\, u_1^2 + \lambda\, v_2^2 + \lambda\, u_3^1 + \lambda\, u_4^3 + (1+\lambda^3 \varepsilon K)\, v^3 &= 0\\
\lambda\, u_1^1 - \lambda\, v_2^1 - \lambda\, u_3^2 + \lambda\, u_4^4 + (1+\lambda^3 \varepsilon K)\, v^4 &= 0
\end{aligned} \tag{3.4.4}$$

where in (3.4.4): $u_k^j = \dfrac{\partial u^j}{\partial x_k}$, $v_k^j = \dfrac{\partial v^j}{\partial x_k}$ $(j,k=1,\dots,4)$, (3.4.4a)

$$K = (u^1)^2 + (u^2)^2 + (v^1)^2 + (v^2)^2 - (u^3)^2 - (u^4)^2 - (v^3)^2 - (v^4)^2 .$$

We construct a differential ideal I, generated by the eight 1-forms associated to (3.4.4) i.e.,

$$\begin{aligned} \alpha(k) &= du^k - \sum_{j=1}^{4} u_j^k \, dx_j \\ \alpha(4+k) &= dv^k - \sum_{j=1}^{4} v_j^k \, dx_j, \end{aligned} \qquad (k = 1,\ldots,4) \qquad (3.4.5)$$

and substitution of u_4^k, v_4^k from (3.4.4) into (3.4.5).

So we consider an ideal I in $\mathbb{R}^{36}$ =

$$\{(x(1),\ldots,x(36))\} = \{(x_1,\ldots,x_4,u^1,\ldots,v^4,u_1^1,\ldots,u_3^1,\ldots,v_1^4,\ldots,v_3^4)\} \quad (3.4.6)$$

generated by (3.4.5), (3.4.4).

We introduce the vector field V

$$V = \sum_{j=1}^{36} F(j) \, \partial_{x(j)} \qquad (3.4.7)$$

The vector field V has to satisfy the condition

$$L_V \, I \subset I, \qquad (3.4.8)$$

which results in an overdetermined system of $8 \times 4 = 32$ partial differential equations for the functions F(j) (j=1,...,36)

From Theorem 1.1.1 we have

F(j) depends only on x(1),...,x(12) (3.4.9)

In [16] we derived the following intermediate results

1: F(1),...,F(4) are independent of x(5),...,x(12)
2: F(5),...,F(12) are linear with respect to x(5),...,x(12). (3.4.10)
3: F(1),...,F(4) are polynomials of degree 3 in x(1),...,x(4).

Combination of the intermediate results (3.4.10) and the conditions resulting from (3.4.8) then leads to the final results [16].

<u>case 1</u> ($\varepsilon=0$, $\lambda^{-1}=0$)

The complete Lie algebra of infinitesimal symmetries of Dirac equations with vanishing rest mass is spanned by 23 generators. Moreover there is a continuous part generated by functions $F^{u^1},\ldots,F^{v^4}$ dependent on $x_1,\ldots,x_4$, which satisfy Dirac equations (3.4.2), due to linearity.

The Lie algebra contains the 15 generators of the conformal group (VF(1),....,VF(15)) and 8 vertical vector fields (VF(16),...,VF(23)).

$$VF(1) := D_{X1}$$

$$VF(2) := D_{X2}$$

$$VF(3) := D_{X3}$$

$$VF(4) := D_{X4}$$

$$VF(5) := 2*D_{X1}*X2 - 2*D_{X2}*X1 - D_{U1}*V1 + D_{U2}*V2 - D_{U3}*V3 + D_{U4}*V4 + D_{V1}*U1 - D_{V2}*U2 + D_{V3}*U3 - D_{V4}*U4$$

$$VF(6) := 2*D_{X1}*X3 - 2*D_{X3}*X1 - D_{U1}*U2 + D_{U2}*U1 - D_{U3}*U4 + D_{U4}*U3 - D_{V1}*V2 + D_{V2}*V1 - D_{V3}*V4 + D_{V4}*V3$$

$$VF(7) := - 2*D_{X2}*X3 + 2*D_{X3}*X2 + D_{U1}*V2 + D_{U2}*V1 + D_{U3}*V4 + D_{U4}*V3 - D_{V1}*U2 - D_{V2}*U1 - D_{V3}*U4 - D_{V4}*U3$$

$$VF(8) := 2*D_{X1}*X4 + 2*D_{X4}*X1 + D_{U1}*U4 + D_{U2}*U3 + D_{U3}*U2 + D_{U4}*U1 + D_{V1}*V4 + D_{V2}*V3 + D_{V3}*V2 + D_{V4}*V1$$

$$VF(9) := 2*D_{X2}*X4 + 2*D_{X4}*X2 + D_{U1}*V4 - D_{U2}*V3 + D_{U3}*V2 - D_{U4}*V1 - D_{V1}*U4 + D_{V2}*U3 - D_{V3}*U2 + D_{V4}*U1$$

$$VF(10) := 2*D_{X3}*X4 + 2*D_{X4}*X3 + D_{U1}*U3 - D_{U2}*U4 + D_{U3}*U1 - D_{U4}*U2 + D_{V1}*V3 - D_{V2}*V4 + D_{V3}*V1 - D_{V4}*V2 \qquad (3.4.11)$$

$$VF(11) := D_{X1}*X1 + D_{X2}*X2 + D_{X3}*X3 + D_{X4}*X4$$

$$VF(12) := (D_{X1}*(X1^2 - X2^2 - X3^2 + X4^2) + 2*D_{X2}*X1*X2 + 2*D_{X3}*X1*X3 + 2*D_{X4}*X1*X4 + D_{U1}*(- 3*X1*U1 + X2*V1 + X3*U2 + X4*U4) + D_{U2}*(- 3*X1*U2 - X2*V2 - X3*U1 + X4*U3) + D_{U3}*(- 3*X1*U3 + X2*V3 + X3*U4 + X4*U2) + D_{U4}*(- 3*X1*U4 - X2*V4 - X3*U3 + X4*U1) + D_{V1}*(- 3*X1*V1 - X2*U1 + X3*V2 + X4*V4) + D_{V2}*(- 3*X1*V2 + X2*U2 - X3*V1 + X4*V3) + D_{V3}*(- 3*X1*V3 - X2*U3 + X3*V4 + X4*V2) + D_{V4}*(- 3*X1*V4 + X2*U4 - X3*V3 + X4*V1))/2$$

$$VF(13) := (2*D_{X1}*X1*X2 + D_{X2}*(- X1^2 + X2^2 - X3^2 + X4^2) + 2*D_{X3}*X2*X3 + 2*D_{X4}*X2*X4 + D_{U1}*(- X1*V1 - 3*X2*U1 + X3*V2 + X4*V4) + D_{U2}*(X1*V2 - 3*X2*U2 + X3*V1 - X4*V3) + D_{U3}*(- X1*V3 - 3*X2*U3 + X3*V4 + X4*V2) + D_{U4}*(X1*V4 - 3*X2*U4 + X3*V3 - X4*V1) + D_{V1}*(X1*U1 - 3*X2*V1 - X3*U2 - X4*U4) + D_{V2}*(- X1*U2 - 3*X2*V2 - X3*U1 + X4*U3) + D_{V3}*(X1*U3 - 3*X2*V3 - X3*U4 - X4*U2) + D_{V4}*(- X1*U4 - 3*X2*V4 - X3*U3 + X4*U1))/2$$

$$VF(14) := (2*D_{X1}*X1*X3 + 2*D_{X2}*X2*X3 + D_{X3}*(- X1^2 - X2^2 + X3^2 + X4^2) + 2*D_{X4}*X3*X4 + D_{U1}*(- X1*U2 - X2*V2 - 3*X3*U1 + X4*U3) + D_{U2}*(X1*U1 - X2*V1 - 3*X3*U2 - X4*U4) + D_{U3}*(- X1*U4 - X2*V4 - 3*X3*U3 + X4*U1) + D_{U4}*(X1*U3 - X2*V3 - 3*X3*U4 - X4*U2) + D_{V1}*(- X1*V2 + X2*U2 - 3*X3*V1 + X4*V3) + D_{V2}*(X1*V1 + X2*U1 - 3*X3*V2 - X4*V4) + D_{V3}*(- X1*V4 + X2*U4 - 3*X3*V3 + X4*V1) + D_{V4}*(X1*V3 + X2*U3 - 3*X3*V4 - X4*V2))/2$$

$$VF(15) := (2*D_{X1}*X1*X4 + 2*D_{X2}*X2*X4 + 2*D_{X3}*X3*X4 + D_{X4}*(X1^2 + X2^2 + X3^2 + X4^2) + D_{U1}*(X1*U4 + X2*V4 + X3*U3 - 3*X4*U1) + D_{U2}*(X1*U3 - X2*V3 - X3*U4 - 3*X4*U2) +$$

$D_{U3}*(X1*U2 + X2*V2 + X3*U1 - 3*X4*U3) + D_{U4}*(X1*U1 - X2*V1 - X3*U2 - 3*X4*U4) + D_{V1}*(X1*V4 - X2*U4 + X3*V3 - 3*X4*V1) + D_{V2}*(X1*V3 + X2*U3 - X3*V4 - 3*X4*V2) + D_{V3}*(X1*V2 - X2*U2 + X3*V1 - 3*X4*V3) + D_{V4}*(X1*V1 + X2*U1 - X3*V2 - 3*X4*V4))/2$

$$
\begin{aligned}
VF(16) &:= D_{U1}*U1 + D_{U2}*U2 + D_{U3}*U3 + D_{U4}*U4 + D_{V1}*V1 + D_{V2}*V2 + D_{V3}*V3 + D_{V4}*V4\\
VF(17) &:= D_{U1}*U2 - D_{U2}*U1 - D_{U3}*U4 + D_{U4}*U3 - D_{V1}*V2 + D_{V2}*V1 + D_{V3}*V4 - D_{V4}*V3\\
VF(18) &:= D_{U1}*U3 + D_{U2}*U4 + D_{U3}*U1 + D_{U4}*U2 + D_{V1}*V3 + D_{V2}*V4 + D_{V3}*V1 + D_{V4}*V2\\
VF(19) &:= D_{U1}*U4 - D_{U2}*U3 - D_{U3}*U2 + D_{U4}*U1 - D_{V1}*V4 + D_{V2}*V3 + D_{V3}*V2 - D_{V4}*V1\\
VF(20) &:= D_{U1}*V1 + D_{U2}*V2 + D_{U3}*V3 + D_{U4}*V4 - D_{V1}*U1 - D_{V2}*U2 - D_{V3}*U3 - D_{V4}*U4\\
VF(21) &:= D_{U1}*V2 - D_{U2}*V1 - D_{U3}*V4 + D_{U4}*V3 + D_{V1}*U2 - D_{V2}*U1 - D_{V3}*U4 + D_{V4}*U3\\
VF(22) &:= D_{U1}*V3 + D_{U2}*V4 + D_{U3}*V1 + D_{U4}*V2 - D_{V1}*U3 - D_{V2}*U4 - D_{V3}*U1 - D_{V4}*U2\\
VF(23) &:= D_{U1}*V4 - D_{U2}*V3 - D_{U3}*V2 + D_{U4}*V1 + D_{V1}*U4 - D_{V2}*U3 - D_{V3}*U2 + D_{V4}*U1
\end{aligned}
\qquad (3.4.11)
$$

The result (3.4.11) is in full agreement with that of Ibragimov (cf. [1])

<u>case 2</u> ($\varepsilon=0$, $\lambda^{-1} \neq 0$)

The complete Lie algebra of infinitesimal symmetries of Dirac equations with nonvanishing rest mass is spanned by 14 generators. Moreover there is a continuous part generated by functions $F^{u^1},\ldots,F^{v^4}$, depending on $x_1,\ldots,x_4$ which have to satisfy Dirac equations with nonvanishing rest mass.

This algebra contains the 10 generators of the Poincaré group VF(1),...,VF(10) and the generators VF(19), VF(20), VF(23), VF(16) (3.4.11).

case 3 ($\varepsilon \neq 0$, $\lambda \neq 0$)

The complete Lie algebra of infinitesimal symmetries of nonlinear Dirac equations with nonvanishing rest mass is spanned by 13 generators.
The generators in this case are the 10 generators of the Poincaré group VF(1),...,VF(10) and VF(19), VF(20), VF(23).

This result generalizes a result of Steeb **et al** [35] where VF(20) was found as an additional symmetry to VF(1),...,VF(10).

case 4 ($\varepsilon \neq 0$, $\lambda = 0$)

The complete Lie algebra in this situation is spanned by 14 generators.
The generators are, the generators of the Poincaré group, VF(1),...,VF(10); VF(19), VF(20), VF(23) and VF(11) - VF(16)/2.

Associated to the infinitesimal symmetries (3.4.11)

$$X^1 = VF(19) = u^4\partial_{u^1} - u^3\partial_{u^2} - u^2\partial_{u^3} + u^1\partial_{u^4} - v^4\partial_{v^1} + v^3\partial_{v^2} + v^2\partial_{v^3} - v^1\partial_{v^4}$$

$$X^2 = VF(20) = v^1\partial_{u^1} + v^2\partial_{u^2} + v^3\partial_{u^3} + v^4\partial_{u^4} - u^1\partial_{v^1} - u^2\partial_{v^2} - u^3\partial_{v^3} - u^4\partial_{v^4}$$

$$X^3 = VF(23) = v^4\partial_{u^1} - v^3\partial_{u^2} - v^2\partial_{u^3} + v^1\partial_{u^4} + u^4\partial_{v^1} - u^3\partial_{v^2} - u^2\partial_{v^3} + u^1\partial_{v^4} \qquad (3.4.12)$$

for nonlinear Dirac Equations (case 3) we shall construct conserved currents in a way analogous to [36].

The generators X^1, X^2, X^3 are vertical vector fields on $J^0(M,N) = \mathbb{R}^{12}$, generated by $\partial_{x_1},\dots,\partial_{x_4}, \partial_{u^1},\dots,\partial_{v^4}$ i.e.,

$$(\pi_M)_* X^j = 0 \qquad (j=1,\dots,3) \qquad (3.4.13)$$

In fact, we need the prolonged vector fields p^1X^1, p^1X^2, p^1X^2 on $J^1(M,N)$, which can be calculated from (3.4.12).

Let $L(\underline{u},\underline{v},\underline{u}_j,\underline{v}_j)$ be the Lagrangian, defined on $J^1(E)$

$$
\begin{aligned}
L = & - u^4v^1_1 + v^4u^1_1 - u^3v^2_1 + v^3u^2_1 - u^2v^3_1 + v^2u^3_1 - u^1v^4_1 + v^1u^4_1 \\
& - v^4v^1_2 - u^4u^1_2 + v^3v^2_2 + u^3u^2_2 - v^2v^3_2 - u^2u^3_2 + v^1v^4_2 + u^1u^4_2 \\
& - u^3v^1_3 + v^3u^1_3 + u^4v^2_3 - v^4v^2_3 - u^1v^3_3 + v^1u^3_3 + u^2v^4_3 + v^2u^4_3 \\
& - u^1v^1_4 + v^1u^1_4 - u^2v^2_4 + v^2u^2_4 - u^3v^3_4 + v^3u^3_4 - u^4v^4_4 + v^4u^4_4 \\
& - K(1+\tfrac{1}{2}\lambda^3\varepsilon K)
\end{aligned}
\qquad (3.4.14)
$$

where $(\underline{x},\underline{u},\underline{v},\underline{u}_j,\underline{v}_j) = (x^1,\ldots,x^4,u^1,\ldots,v^4,u^1_1,\ldots,u^1_4,\ldots,v^4_1,\ldots,v^4_4)$ are local coordinates on $J^1(M,N) = \mathbb{R}^{44}$.

An easy calculation shows that the Euler-Lagrange equations associated to (3.4.14) i.e.,

$$\frac{\partial}{\partial x^a}\frac{\partial L}{\partial z^A_a} - \frac{\partial L}{\partial z^A} = 0, \qquad (3.4.15)$$

are just nonlinear Dirac equations (3.4.4).

In (3.4.15) we used z^A $(A=1,\ldots,8)$ instead of $u^1,\ldots,u^4,v^1,\ldots,v^4$ and summation convention $(A=1,\ldots,8;a=1,\ldots,4)$ if an index occurs twice.

We introduce the Cartan form Θ defined by [34]

$$\Theta = L\omega + (\partial^a_A L)\theta^A\omega_a, \qquad (3.4.16)$$

where

$$
\begin{aligned}
\omega &= dx^1dx^2dx^3dx^4, \\
\partial_a &= \partial_{x^a},\ \partial_A = \partial_{z^A},\ \partial^a_A = \partial_{z^A_a}, \qquad (3.4.16a) \\
\omega_a &= \partial_a \lrcorner\ \omega, \\
\theta^A &= dz^A - z^A_a\, dx^a.
\end{aligned}
$$

(z^A_a referring to either u^j_a or v^j_a).

From (3.4.16) we derive

$$\Theta = L\omega + (\partial^a_A L)(dz^A)\omega_a - (\partial^a_A L)\, z^A_a \omega \tag{3.4.17}$$

$$= \{L - (\partial^a_A L) z^A_a\}\omega + (\partial^a_A L)(dz^A)\omega_a .$$

Since L, defined by (3.4.14) is linear in z^A_a we derive

$$L - (\partial^a_A L)\, z^A_a = -K(1 + \tfrac{1}{2}\lambda^3 \varepsilon K). \tag{3.4.18}$$

We now want to compute

$$(p^1X^i)\Theta \; ,$$

i.e., the Lie derivative of the Cartan form Θ by the vector field p^1X^i (i=1,...,3),

We prove the following

Lemma 3.4.1

$$(p^1X^i)\;\Theta = 0 \quad (i=1,...,3)$$

Proof The proof is in two parts

$$\underline{1^o} : p^1X^i\{(-K(1+\tfrac{1}{2}\lambda^3\varepsilon K)\omega\} = 0 \qquad (i=1,...,3) \tag{3.4.19a}$$

$$\underline{2^o} : p^1X^i\{(\partial^a_A L)dz^A\} = 0 \qquad (i=1,...,3;\ a=1,...,4) \tag{3.4.19b}$$

Proof of 1^o

$$(p^1X^i)\{-K(1+\tfrac{1}{2}\lambda^3\varepsilon K)\omega\} =$$

$$= [(p^1X^i)(-K(1+\tfrac{1}{2}\lambda^3\varepsilon K))]\omega$$

$$= (p^1X^i)\lrcorner\,(-1-\lambda^3\varepsilon K)dK\omega$$

and due to definition of K (3.4.4a)

$$dK = 2(u^1du^1 + u^2du^2 - u^3du^3 - u^4du^4 + v^1dv^1 + v^2dv^2 - v^3dv^3 - v^4dv^4)$$

an easy calculation leads to

$$(p^1X^i) \lrcorner\, dK = 0, \quad (i= 1,\dots,3)$$

which completes the proof of part 1^o.

Proof of 2^o

In order to prove (4.19b) we introduce four 1-forms

$$\begin{aligned}
V_1^* &= (\partial_A^1 L)dz^A = \ v^4du^1 + v^3du^2 + v^2du^3 + v^1du^4 - u^4dv^1 - v^3dv^2 - u^2dv^3 - u^1dv^4 \\
V_2^* &= (\partial_A^2 L)dz^A = -u^4du^1 + u^3du^2 - u^2du^3 + u^1du^4 - v^4dv^1 + v^3dv^2 - v^2dv^3 + v^1dv^4 \\
V_3^* &= (\partial_A^3 L)dz^A = \ v^3du^1 - v^4du^2 + v^1du^3 - v^2du^4 - u^3dv^1 + u^4dv^2 - u^1dv^3 + u^2dv^4 \\
V_4^* &= (\partial_A^4 L)dz^A = \ v^1du^1 + v^2du^2 + v^3du^3 + v^4du^4 - u^1dv^1 - u^2dv^2 - u^3dv^3 - u^4dv^4
\end{aligned} \qquad (3.4.20)$$

from which we obtain

$$\begin{aligned}
dV_1^* &= -2[du^1dv^4 + du^2dv^3 + du^3dv^2 + du^4dv^1] \\
dV_2^* &= +2[du^1du^4 - du^2du^3 + dv^1dv^4 - dv^2dv^3] \\
dV_3^* &= +2[-du^1dv^3 + du^2dv^4 - du^3dv^1 + du^4dv^2] \\
dv_4^* &= -2[du^1dv^1 + du^2dv^2 + du^3dv^3 + du^4dv^4]
\end{aligned} \qquad (3.4.20a)$$

Using (3.4.12), (3.4.20), (3.4.20a) a somewhat lengthy calculation leads to the result (3.4.19a) i.e.

$$p^1X^i\{V_j^*\} = 0 \quad (i= 1,\dots,3;\ j= 1,\dots,4) \qquad (3.4.21)$$

this completes the proof of Lemma 3.4.1.

Now due to the relation □

$$(p^1X^i)\Theta = (p^1X^i) \lrcorner\, d\Theta + d(p^1X^i \lrcorner\, \Theta) = 0 \quad (i= 1,..,3)$$

and (cf. [36])

$$(p^1X^i) \lrcorner d\Theta = 0 \text{ "mod I"} \quad (i= 1,\ldots,3)$$

we arrive at

$$d(p^1X^i \lrcorner \Theta) = 0 \text{ "mod I"} \quad (i= 1,\ldots,3) \tag{3.4.22}$$

i.e.,

$$p^1X^i \lrcorner \Theta \text{ is a conserved current } (i=1,\ldots,3). \tag{3.4.23}$$

Combination of (3.4.12), (3.4.16), (3.4.21) leads to

$$p^1X^i \lrcorner \Theta = (p^1X^i \lrcorner V_a^*)\omega_a$$

i.e.,

the conserved currents associated to X^1, X^2, X^3 are given by

$$
\begin{aligned}
1:\quad & +2(u^4v^4 - u^3v^3 - u^2v^2 + u^1v^1)dx_2dx_3dx_4 \\
& -((u^1)^2 + (u^2)^2 - (u^3)^2 - (u^4)^2 - (v^1)^2 - (v^2)^2 + (v^3)^2 + v^4)^2)\, dx_1dx_3dx_4 \\
& +2(u^4v^3 + u^3v^4 - u^2v^1 - u^1v^2)dx_1dx_2dx_4 \\
& -2(u^4v^1 - u^3v^2 - u^2v^3 + u^1v^4)dx_1dx_2dx_3 \\
2:\quad & +2(v^1v^4 + v^2v^3 + u^1u^4 + u^2u^3)dx_2dx_3dx_4 \\
& -2(-u^4v^1 + u^3v^2 - u^2v^3 + u^1v^4)dx_1dx_3dx_4 \\
& +2(v^1v^3 - v^2v^4 + u^1u^3 - u^2u^4)dx_1dx_2dx_4 \\
& -((u^1)^2 + (u^2)^2 + (u^3)^2 + (u^4)^2 + (v^1)^2 + (v^2)^2 + (v^3)^2 + (v^4)^2)dx_1dx_2dx_3 \\
3:\quad & (-(u^1)^2 + (u^2)^2 + (u^3)^2 - (u^4)^2 + (v^1)^2 - (v^2)^2 - (v^3)^2 + (v^4)^2)dx_2dx_3dx_4 \\
& -2(u^4v^4 - u^3v^3 + u^2v^2 + u^1v^1)dx_1dx_3dx_4 \\
& +2(v^3v^4 - v^2v^1 - u^3u^4 + u^1u^2)dx_1dx_2dx_4 \\
& -2(v^1v^4 - v^3v^2 - u^1u^4 + u^2u^3)dx_1dx_2dx_3
\end{aligned}
\tag{3.4.24}
$$

3.5 Infinitesimal symmetries of self dual SU(2) Yang-Mills equations. The Belavin-Polyakov-Schwartz-Tyupkin instanton and the monopole solution

First of all we shall give a very short description of the SU(2)-gauge theory. For a more extensive elementary exposition we refer to the survey paper by M.K. Prasad [32], from which we adopt the notations. Then we shall indicate how to derive the Lie algebra of infinitesimal symmetries of the self dual SU(2) Yang-Mills equations. Using these results we demonstrate the way in which the BPST-instanton solution can be obtained as a similarity solution [3]. By imposing additional conditions we compute the Lie algebra of infinitesimal symmetries of the static self dual SU(2) equations. Analogous to the construction of the BSPT-instanton we derive the ansatz [31] leading to the monopole solution.

Let M be a 4-dimensional Euclidean space with coordinates $x_\mu = (x_1,\dots,x_4)$ so there will be no distinction between contravariant and covariant indices, $x_\mu = x^\mu$. The basic object in gauge theory is the Yang-Mills gauge potential. The gauge potential is a set of fields A_μ^a $(a=1,\dots,3;\mu=1,\dots,4)$
It is convenient to introduce a matrix valued vector field $A_\mu(x)$, by,

$$A_\mu = gT^a A_\mu^a \ ; \ T^a = \frac{\sigma^a}{2i} \ (a=1,2,3), \qquad (3.5.1)$$

where σ^a are the Pauli matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \ \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , \ \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ ; \qquad (3.5.1a)$$

g being a constant called the gauge coupling constant. Throughout this section we shall use the Einstein summation convention when an index occurs twice. (a,b,... take on the values 1,2,3; μ,ν take on the values 1,...,4)
From the matrix valued gauge potential $A_\mu \, dx_\mu$ one constructs the matrix valued field strength $F_{\mu\nu}(x)$ by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \qquad (3.5.2)$$

where $\partial_\mu = \partial_{x_\mu}$; $[A_\mu, A_\nu] = A_\mu A_\nu - A_\nu A_\mu$.

If one defines the covariant derivative

$$D_\mu = \partial_\mu + A_\mu \tag{3.5.3}$$

then (3.5.2) is rewritten as

$$F_{\mu\nu} = [D_\mu, D_\nu] \tag{3.5.4}$$

In explicit component form,

$$F_{\mu\nu} = gT^a F^a_{\mu\nu}, \tag{3.5.5a}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon_{abc} A^b_\mu\, A^c_\nu . \tag{3.5.5b}$$

and $$\varepsilon_{abc} = \begin{cases} 1 \text{ if abc is an even permutation of } 1\ 2\ 3 \\ -1 \text{ if abc is an odd permutation of } 1\ 2\ 3 \\ 0 \text{ otherwise.} \end{cases}$$

We will use the expression "static gauge fields" to refer to gauge potentials that are independent of x_4 (x_4 to be considered as time) i.e.,

$$\partial_4 A_\mu(x) = 0 \qquad (\mu = 1, \ldots, 4) \tag{3.5.6}$$

For gauge potentials that depend on all four coordinates $x_1, \ldots, x_4$ the action functional is defined by

$$S = \frac{1}{4} \int F^a_{\mu\nu}\, F^a_{\mu\nu}\, (d^4x), \tag{3.5.7a}$$

the integral taken over $\mathbb{R}^4$, while for static gauge fields we define the energy functional by

$$E = \frac{1}{4} \int F^a_{\mu\nu} F^a_{\mu\nu} (d^3x) \tag{3.5.7b}$$

whereas in (3.5.7b) the integral is taken over $\mathbb{R}^3$.

The extremals of the action S (or of the energy E for static gauge fields) are found by standard calculus of variations techniques leading to the Euler-Lagrange equations

$$\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0 = [D_\mu, F_{\mu\nu}] \tag{3.5.8}$$

or, in components

$$\partial_\mu F^a_{\mu\nu} + g\varepsilon_{abc}\, A^b_\mu F^c_{\mu\nu} = 0 \tag{3.5.8a}$$

(3.5.8a) is a system of <u>second order</u> nonlinear partial differential equations for the 12 unknown functions $A^a_\mu(x)$ $(a=1,\dots,3;\ \mu=1,\dots,4)$ and seems hard to solve.

Then one introduces the dual gauge field strength $*F_{\mu\nu}$ defined as

$$*F_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\lambda\rho}\, F_{\lambda\rho} \tag{3.5.9}$$

where $\varepsilon_{\mu\nu\lambda\rho}$ is the completely antisymmetric tensor in M, defined by

$$\varepsilon_{\mu\nu\lambda\rho} = \begin{cases} +\ 1 \text{ if } \mu\nu\lambda\rho \text{ is an even permutation of } 1\ 2\ 3\ 4 \\ -\ 1 \text{ if } \mu\nu\lambda\rho \text{ is an odd permutation of } 1\ 2\ 3\ 4 \\ \ \ 0 \text{ otherwise.} \end{cases}$$

Since D_μ (3.5.3) satisfies the Jacobi identity

$$[D_\lambda,[D_\mu,D_\nu]] + [D_\mu,[D_\nu,D_\lambda]] + [D_\nu,[D_\lambda,D_\mu]] = 0 \tag{3.5.10}$$

multiplication of (3.5.10) by $\varepsilon_{\mu\nu\lambda\rho}$ results in

$$[D_\mu, *F_{\mu\nu}] = 0. \tag{3.5.11}$$

If we compare (3.5.8), (3.5.11) we see that any gauge field which is self dual

$$*F_{\mu\nu} = F_{\mu\nu}, \tag{3.5.12}$$

automatically satisfies (3.5.6).
(3.5.9) is a system of <u>first order</u> nonlinear partial differential equations. Instanton solutions, and monopole solutions for the static gauge fields satisfy (3.5.12) [32] and the condition that (3.5.7a), (3.5.7b) are finite.

Written out, (3.5.12) takes the form

$$F_{12} = F_{34},\ F_{13} = -F_{24},\ F_{14} = F_{23}\ . \tag{3.5.13}$$

So in components the self dual Yang-Mills equations is described as a system of 9 nonlinear partial differential equations,

$$\begin{aligned}
&-A^1_{4,1} + A^1_{3,2} - A^1_{2,3} + A^1_{1,4} + g\{-A^2_1A^3_4 + A^2_2A^3_3 - A^2_3A^3_2 + A^2_4A^3_1\} = 0\\
&-A^2_{4,1} + A^2_{3,2} - A^2_{2,3} + A^2_{1,4} + g\{\ A^1_1A^3_4 - A^1_2A^3_3 + A^1_3A^3_2 - A^1_4A^3_1\} = 0\\
&-A^3_{4,1} + A^3_{3,2} - A^3_{2,3} + A^3_{1,4} + g\{-A^1_1A^2_4 + A^1_2A^2_3 - A^1_3A^2_3 + A^1_4A^2_1\} = 0\\
&\ A^1_{3,1} + A^1_{4,2} - A^1_{1,3} - A^1_{2,4} + g\{\ A^2_1A^3_3 + A^2_2A^3_4 - A^2_3A^3_1 - A^2_4A^3_2\} = 0\\
&\ A^2_{3,1} + A^2_{4,2} - A^2_{1,3} - A^2_{2,4} + g\{-A^1_1A^3_3 + A^1_2A^3_4 + A^1_3A^3_1 + A^1_4A^3_2\} = 0\\
&\ A^3_{3,1} + A^3_{4,2} - A^3_{1,3} - A^3_{2,4} + g\{\ A^1_1A^2_3 + A^1_2A^2_4 - A^1_3A^2_1 - A^1_4A^2_2\} = 0\\
&\ A^1_{2,1} - A^1_{1,2} - A^1_{4,3} + A^1_{3,4} + g\{\ A^2_1A^3_2 - A^2_2A^3_1 - A^2_3A^3_4 + A^2_4A^3_3\} = 0\\
&\ A^2_{2,1} - A^2_{1,2} - A^2_{4,3} + A^2_{3,4} + g\{-A^1_1A^3_2 + A^1_2A^3_1 + A^1_3A^3_4 - A^1_4A^3_3\} = 0\\
&\ A^3_{2,1} - A^3_{1,2} - A^3_{4,3} + A^3_{3,4} + g\{\ A^1_1A^2_2 - A^1_2A^2_1 - A^1_3A^2_4 + A^1_4A^2_3\} = 0
\end{aligned} \tag{3.5.14}$$

whereas in (3.5.14)

$$A^a_{\mu,\nu} = \partial_\nu A^a_\mu \quad (a = 1,\dots,3;\ \mu,\nu=1,\dots,4)$$

We construct a differential ideal I defined on a 55-dimensional space

$$\mathbb{R}^{55} = \{(x_1,x_2,x_3,x_4,\ A^1_1\ ,\dots,A^1_4,\dots,A^1_{1,1},\dots,A^3_{4,4},\hat{A}^1_{1,4},\dots,\hat{A}^3_{3,4})\}, \tag{3.5.16}$$

generated by the twelve 1-forms

$$\alpha^a_\mu = dA^a_\mu - A^a_{\mu,\nu}dx_\nu, \quad (a=1,\dots,3;\ \mu=1,\dots,4) \tag{3.5.17}$$

and substitution of $A^1_{1,4},\dots,A^3_{3,4}$, obtained by solving (3.5.14) for these variables. (in (3.5.16) $\hat{A}^1_{1,4},\dots,$ denotes deletion)

The symmetry condition

$$\mathcal{L}_V \, I \subset I,$$

where V is a vector field defined on $\mathbb{R}^{55}$ results in an overdetermined system of 48 partial differential equations for the components of this vector field. Since the most general Lie contact symmetries are Lie point symmetries [1], the (16) components of $\partial_1,\dots,\partial_4,\partial_{A^1_1},\dots,\partial_{A^3_4}$ are functions, depending on $x_1,\dots,x_4,\ A^1_1,\dots,A^3_4$.

The general solution of this overdetermined system constitutes a Lie algebra of infinitesimal symmetries, generated by the vector fields

```
VF(1) := (D_A11*DF(F(71),X1) + D_A12*DF(F(71),X2) + D_A13*DF(F(71),
         X3) + D_A14*DF(F(71),X4) + D_A21*A31*GE*F(71) + D_A22*GE*
         A32*F(71) + D_A23*GE*A33*F(71) + D_A24*A34*GE*F(71) -
         D_A31*A21*GE*F(71) - D_A32*GE*A22*F(71) - D_A33*GE*A23*F(
         71) - D_A34*A24*GE*F(71))/GE
VF(2) := ( - D_A11*A21*GE*F(81) - D_A12*GE*A22*F(81) - D_A13*GE*A23*
         F(81) - D_A14*A24*GE*F(81) + D_A21*A11*GE*F(81) + D_A22*GE      (3.5.18)
         *A12*F(81) + D_A23*GE*A13*F(81) + D_A24*GE*A14*F(81) -
         D_A31*DF(F(81),X1) - D_A32*DF(F(81),X2) - D_A33*DF(F(81),
         X3) - D_A34*DF(F(81),X4))/GE
VF(3) := (D_A11*A31*GE*F(63) + D_A12*GE*A32*F(63) + D_A13*GE*A33*F(
         63) + D_A14*A34*GE*F(63) - D_A21*DF(F(63),X1) - D_A22*DF(F
         (63),X2) - D_A23*DF(F(63),X3) - D_A24*DF(F(63),X4) -
         D_A31*A11*GE*F(63) - D_A32*GE*A12*F(63) - D_A33*GE*A13*F(
         63) - D_A34*GE*A14*F(63))/GE
```

$$VF(4) := D_{X1}$$

$$VF(5) := D_{X2}$$

$$VF(6) := D_{X3}$$

$$VF(7) := D_{X4}$$

$$VF(8) := D_{X1}*X2 - D_{X2}*X1 + D_{A11}*A12 - D_{A12}*A11 + D_{A21}*A22 - D_{A22}*A21 + D_{A31}*A32 - D_{A32}*A31$$

$$VF(9) := - D_{X1}*X3 + D_{X3}*X1 - D_{A11}*A13 + D_{A13}*A11 - D_{A21}*A23 + D_{A23}*A21 - D_{A31}*A33 + D_{A33}*A31$$

$$VF(10) := - D_{X1}*X4 + D_{X4}*X1 - D_{A11}*A14 + D_{A14}*A11 - D_{A21}*A24 + D_{A24}*A21 - D_{A31}*A34 + D_{A34}*A31$$

$$VF(11) := - D_{X2}*X3 + D_{X3}*X2 - D_{A12}*A13 + D_{A13}*A12 - D_{A22}*A23 + D_{A23}*A22 - D_{A32}*A33 + D_{A33}*A32$$

$$VF(12) := D_{X2}*X4 - D_{X4}*X2 + D_{A12}*A14 - D_{A14}*A12 + D_{A22}*A24 - D_{A24}*A22 + D_{A32}*A34 - D_{A34}*A32$$

$$VF(13) := - D_{X3}*X4 + D_{X4}*X3 - D_{A13}*A14 + D_{A14}*A13 - D_{A23}*A24 + D_{A24}*A23 - D_{A33}*A34 + D_{A34}*A33$$

$$\begin{aligned} VF(14) := {} & D_{X1}*X1 + D_{X2}*X2 + D_{X3}*X3 + D_{X4}*X4 - D_{A11}*A11 - D_{A12}*A12 \\ & - D_{A13}*A13 - D_{A14}*A14 - D_{A21}*A21 - D_{A22}*A22 - D_{A23}*A23 \\ & - D_{A24}*A24 - D_{A31}*A31 - D_{A32}*A32 - D_{A33}*A33 - D_{A34}*A34 \end{aligned}$$

$$\begin{aligned} VF(15) := {} & D_{X1}*(X4^2 + X3^2 + X2^2 - X1^2) - 2*D_{X2}*X2*X1 - 2*D_{X3}*X3*X1 \\ & - 2*D_{X4}*X4*X1 + 2*D_{A11}*(X4*A14 + X3*A13 + X2*A12 + X1*A11) + 2*D_{A12}*(- X2*A11 + X1*A12) \\ & + 2*D_{A13}*(- X3*A11 + X1*A13) + 2*D_{A14}*(- X4*A11 + X1*A14) \\ & + 2*D_{A21}*(A21*X1 + X4*A24 + X3*A23 + X2*A22) + 2*D_{A22}*(- A21*X2 + X1*A22) \\ & + 2*D_{A23}*(- A21*X3 + X1*A23) + 2*D_{A24}*(- A21*X4 + X1*A24) \\ & + 2*D_{A31}*(X4*A34 + X3*A33 + X2*A32 + X1*A31) + 2*D_{A32}*(- X2*A31 + X1*A32) \\ & + 2*D_{A33}*(- X3*A31 + X1*A33) + 2*D_{A34}*(- X4*A31 + X1*A34) \end{aligned}$$

$$\begin{aligned}VF(16) := & -2*D_{X1}*X2*X1 + D_{X2}*(X4^{2} + X3^{2} - X2^{2} + X1^{2}) - 2*D_{X3}*X3*X2 - 2*D_{X4}*X4*X2 + 2*D_{A11}*(X2*A11 - X1*A12) \\ & + 2*D_{A12}*(X4*A14 + X3*A13 + X2*A12 + X1*A11) + 2*D_{A13}*(- X3*A12 + X2*A13) + 2*D_{A14}*(- X4*A12 + X2*A14) \\ & + 2*D_{A21}*(A21*X2 - X1*A22) + 2*D_{A22}*(A21*X1 + X4*A24 + X3*A23 + X2*A22) + 2*D_{A23}*(- X3*A22 + X2*A23) \\ & + 2*D_{A24}*(- X4*A22 + X2*A24) + 2*D_{A31}*(X2*A31 - X1*A32) + 2*D_{A32}*(X4*A34 + X3*A33 + X2*A32 + X1*A31) \\ & + 2*D_{A33}*(- X3*A32 + X2*A33) + 2*D_{A34}*(- X4*A32 + X2*A34)\end{aligned}$$

$$\begin{aligned}VF(17) := & -2*D_{X1}*X3*X1 - 2*D_{X2}*X3*X2 + D_{X3}*(X4^{2} - X3^{2} + X2^{2} + X1^{2}) - 2*D_{X4}*X4*X3 + 2*D_{A11}*(X3*A11 - X1*A13) \\ & + 2*D_{A12}*(X3*A12 - X2*A13) + 2*D_{A13}*(X4*A14 + X3*A13 + X2*A12 + X1*A11) + 2*D_{A14}*(- X4*A13 + X3*A14) \\ & + 2*D_{A21}*(A21*X3 - X1*A23) + 2*D_{A22}*(X3*A22 - X2*A23) + 2*D_{A23}*(A21*X1 + X4*A24 + X3*A23 + X2*A22) \\ & + 2*D_{A24}*(- X4*A23 + X3*A24) + 2*D_{A31}*(X3*A31 - X1*A33) + 2*D_{A32}*(X3*A32 - X2*A33) \\ & + 2*D_{A33}*(X4*A34 + X3*A33 + X2*A32 + X1*A31) + 2*D_{A34}*(- X4*A33 + X3*A34)\end{aligned}$$

$$\begin{aligned}VF(18) := & -2*D_{X1}*X4*X1 - 2*D_{X2}*X4*X2 - 2*D_{X3}*X4*X3 + D_{X4}*(- X4^{2} + X3^{2} + X2^{2} + X1^{2}) + 2*D_{A11}*(X4*A11 - X1*A14) \\ & + 2*D_{A12}*(X4*A12 - X2*A14) + 2*D_{A13}*(X4*A13 - X3*A14) + 2*D_{A14}*(X4*A14 + X3*A13 + X2*A12 + X1*A11) \\ & + 2*D_{A21}*(A21*X4 - X1*A24) + 2*D_{A22}*(X4*A22 - X2*A24) + 2*D_{A23}*(X4*A23 - X3*A24) \\ & + 2*D_{A24}*(A21*X1 + X4*A24 + X3*A23 + X2*A22) + 2*D_{A31}*(X4*A31 - X1*A34) + 2*D_{A32}*(X4*A32 - X2*A34) \\ & + 2*D_{A33}*(X4*A33 - X3*A34) + 2*D_{A34}*(X4*A34 + X3*A33 + X2*A32 + X1*A31)\end{aligned}$$

In (3.5.18) we introduced the following notations

$$\text{Aa}\mu = A^a_\mu \; ; \; D = \partial \; ; \; X1 = X_1 \; ; \; X2 = X_2 \; ; \; X3 = X_3 \; ; \; X4 = X_4, \qquad (3.5.19)$$

while in VF(1), VF(2), VF(3) the functions F(63), F(71), F(81) are arbitrary, depending on x_1,x_2,x_3,x_4.
VF(1), VF(2), VF(3) are just the generators of the gauge transformations [32]. The vector fields VF(4), VF(5), VF(6), VF(7) are generators of translations while VF(8),...,VF(13) refer to rotations; VF(4),...,VF(18) are the generators of the conformal group.

To construct similarity solutions associated to infinitesimal symmetries of self dual Yang-Mills equations (3.5.14) we start from the vector fields X_1, X_2, X_3,

$$\begin{aligned} X_1 &= VF(8) + F(1)*VF(1) + F(2)*VF(2) + F(3)*VF(3) \\ X_2 &= VF(9) + F(4)*VF(1) + F(5)*VF(2) + F(6)*VF(3) \\ X_3 &= VF(10) + F(7)*VF(1) + F(8)*VF(2) + F(9)*VF(3) \end{aligned} \qquad (3.5.20)$$

i.e., we take a combination of a rotation and a gauge transformation. (In (3.5.20), the meaning of F(1)*VF(1) is: take F(71) = F(1) in VF(1)). We also construct the commutators of the vector fields X_1,X_2,X_3

$$[X_1,X_2], \; [X_1,X_3] \; , \; [X_2,X_3], \qquad (3.5.21)$$

and make the following choice for the functions F(i) (i=1,...,9)

$$\begin{aligned} &F(1) = 0, \; F(2) = -1, \; F(3) = 0, \\ &F(4) = 0, \; F(5) = 0, \; F(6) = -1 \\ &F(7) = -1, \; F(8) = 0, \; F(9) = 0 \end{aligned} \qquad (3.5.22)$$

Now the contraction of the vector fields X_1, X_2, X_3, $[X_1,X_2]$, $[X_1,X_3]$ $[X_2,X_3]$, (3.5.20) (3.5.21) (3.5.22) and the 12 contact 1-forms (3.5.17) results in 72 partial differential equations for the functions A^a_μ $(a=1,\ldots,3;\mu=1,\ldots,4)$
The set of 72 equations is given by

```
1:    0 =   A12 + A21 - A11,1*X2 + A11,2*X1
2:    0 =   A22 - A11 - A21,1*X2 + A21,2*X1
3:    0 =   A32 - A31,1*X2 + A31,2*X1
4:    0 =   A22 - A11 - A12,1*X2 + A12,2*X1
5:    0 = - A12 - A21 - A22,1*X2 + A22,2*X1
6:    0 = - A31 - A32,1*X2 + A32,2*X1
7:    0 =   A23 - A13,1*X2 + A13,2*X1
8:    0 = - A13 - A23,1*X2 + A23,2*X1
9:    0 = - A33,1*X2 + A33,2*X1
10:   0 =   A24 - A14,1*X2 + A14,2*X1
11:   0 = - A14 - A24,1*X2 + A24,2*X1
12:   0 = - A34,1*X2 + A34,2*X1
13:   0 = - A13 - A31 + A11,1*X3 - A11,3*X1
14:   0 = - A23 + A21,1*X3 - A21,3*X1
15:   0 = - A33 + A11 + A31,1*X3 - A31,3*X1
16:   0 = - A32 + A12,1*X3 - A12,3*X1
17:   0 =   A22,1*X3 - A22,3*X1
18:   0 =   A12 + A32,1*X3 - A32,3*X1
19:   0 = - A33 + A11 + A13,1*X3 - A13,3*X1
20:   0 =   A21 + A23,1*X3 - A23,3*X1
21:   0 =   A13 + A31 + A33,1*X3 - A33,3*X1
22:   0 = - A34 + A14,1*X3 - A14,3*X1
23:   0 =   A24,1*X3 - A24,3*X1
24:   0 =   A14 + A34,1*X3 - A34,3*X1                (3.5.23)
25:   0 = - A14 + A11,1*X4 - A11,4*X1
26:   0 = - A24 - A31 + A21,1*X4 - A21,4*X1
27:   0 = - A31,4*X1 - A34 + A21 + A31,1*X4
28:   0 = - A12,4*X1 + A12,1*X4
29:   0 = - A22,4*X1 - A32 + A22,1*X4
30:   0 = - A32,4*X1 + A22 + A32,1*X4
31:   0 = - A13,4*X1 + A13,1*X4
32:   0 = - A23,4*X1 - A33 + A23,1*X4
33:   0 = - A33,4*X1 + A23 + A33,1*X4
34:   0 =   A11 - A14,4*X1 + A14,1*X4
35:   0 = - A34 + A21 - A24,4*X1 + A24,1*X4
36:   0 =   A24 + A31 - A34,4*X1 + A34,1*X4
37:   0 =   A11,2*X3 - A11,3*X2
38:   0 = - A31 + A21,2*X3 - A21,3*X2
39:   0 =   A21 + A31,2*X3 - A31,3*X2
40:   0 = - A13 + A12,2*X3 - A12,3*X2
41:   0 = - A23 - A32 + A22,2*X3 - A22,3*X2
42:   0 = - A33 + A22 + A32,2*X3 - A32,3*X2
43:   0 =   A12 + A13,2*X3 - A13,3*X2
44:   0 = - A33 + A22 + A23,2*X3 - A23,3*X2
45:   0 =   A23 + A32 + A33,2*X3 - A33,3*X2
46:   0 =   A14,2*X3 - A14,3*X2
47:   0 = - A34 + A24,2*X3 - A24,3*X2
48:   0 =   A24 + A34,2*X3 - A34,3*X2
49:   0 =   A31 + A11,2*X4 - A11,4*X2
```

```
50:    0 =   A21,2*X4 - A21,4*X2
51:    0 = - A31,4*X2 - A11 + A31,2*X4
52:    0 = - A12,4*X2 - A14 + A32 + A12,2*X4
53:    0 = - A22,4*X2 - A24 + A22,2*X4
54:    0 = - A32,4*X2 - A34 - A12 + A32,2*X4
55:    0 = - A13,4*X2 + A33 + A13,2*X4
56:    0 = - A23,4*X2 + A23,2*X4
57:    0 = - A33,4*X2 - A13 + A33,2*X4
58:    0 =   A34 + A12 - A14,4*X2 + A14,2*X4
59:    0 =   A22 - A24,4*X2 + A24,2*X4
60:    0 = - A14 + A32 - A34,4*X2 + A34,2*X4
61:    0 =   A21 - A11,3*X4 + A11,4*X3                    (3.5.23)
62:    0 = - A11 - A21,3*X4 + A21,4*X3
63:    0 =   A31,4*X3 - A31,3*X4
64:    0 =   A12,4*X3 + A22 - A12,3*X4
65:    0 =   A22,4*X3 - A12 - A22,3*X4
66:    0 =   A32,4*X3 - A32,3*X4
67:    0 =   A13,4*X3 + A14 + A23 - A13,3*X4
68:    0 =   A23,4*X3 + A24 - A13 - A23,3*X4
69:    0 =   A34 + A33,4*X3 - A33,3*X4
70:    0 =   A24 - A13 + A14,4*X3 - A14,3*X4
71:    0 = - A14 - A23 + A24,4*X3 - A24,3*X4
72:    0 = - A33 + A34,4*X3 - A34,3*X4
```

where in (3.5.23) we used the same notation as introduced in (3.5.19) and

$$\mathrm{A}a\mu,\nu = \partial_\nu A^a_\mu \tag{3.5.23a}$$

The system can be solved in a straight-forward way, leading to the following result

$$\begin{array}{llll} A^1_1 = x_4 f(r) & A^1_2 = x_3 f(r) & A^1_3 = -x_2 f(r) & A^1_4 = -x_1 f(r) \\ A^2_1 = -x_3 f(r) & A^2_2 = x_4 f(r) & A^2_3 = x_1 f(r) & A^2_4 = -x_2 f(r) \\ A^3_1 = x_2 f(r) & A^3_2 = -x_1 f(r) & A^3_3 = x_4 f(r) & A^3_4 = -x_3 f(r) \end{array} \tag{3.5.24}$$

where

$$r = (x_1^2+x_2^2+x_3^2+x_4^2)^{\frac{1}{2}}. \tag{3.5.24a}$$

At the derivation of the monopole solution, we shall discuss in some more detail how to solve a system of partial differential equations like (3.5.24). Substitution of (3.5.25) into (3.5.14) yields an ordinary differential equation for the function f(r)

$$\frac{df(r)}{dr} + rgf(r)^2 = 0 \; ; \qquad (3.5.25)$$

the solution of this equation is given by

$$f(r) = + \left(\tfrac{1}{2}\, g\right)^{-1} \frac{1}{r^2+C} \qquad \text{(C a constant)} \qquad (3.5.26)$$

The result (3.5.24), (3.5.26) is just the Belavin-Polyakov-Schwartz-Tyupkin instanton solution [3].

More general, if we choose

$$F(2) = \pm 1, \; F(6) = \pm 1, \; F(7) = \pm 1 \qquad (3.5.27)$$

and

$$F(2)*F(6)*F(7) = -1, \qquad (3.5.28)$$

or equivalently,

$$F(7) = -F(2)*F(6) \qquad (3.5.28a)$$

we arrive at

$$\begin{array}{llll} A_1^1 = x_4 f(r) & A_2^1 = x_3 f(r) & A_3^1 = -x_2 f(r) & A_4^1 = -x_1 f(r) \\ A_1^2 = x_3 f(r)F(2) & A_2^2 = -x_4 f(r)F(2) & A_3^2 = -x_1 f(r)F(2) & A_4^2 = x_2 f(r)F(2) \\ A_1^3 = -x_2 f(r)F(6) & A_2^3 = x_1 f(r)F(6) & A_3^3 = -x_4 f(r)F(6) & A_4^3 = x_3 f(r)F(6) \end{array} \qquad (3.5.29)$$

while f(r) has to satisfy (3.5.14), which results in

$$\frac{df(r)}{dr} + r\, g(F(2)*F(6))f(r)^2 = 0. \qquad (3.5.29a)$$

Choosing F(2), F(6), F(7) as in (3.5.27) and

$$F(2)*F(6)*F(7) = +1 \qquad (3.5.30)$$

the result is

$$
\begin{array}{llll}
A_1^1 = x_4 f(r) & A_2^1 = -x_3 f(r) & A_3^1 = x_2 f(r) & A_4^1 = -x_1 f(r) \\
A_1^2 = -x_3 f(r)F(2) & A_2^2 = -x_4 f(r)F(2) & A_3^2 = x_1 f(r)F(2) & A_4^2 = x_2 f(r)F(2) \\
A_1^3 = x_2 f(r)F(6) & A_2^3 = -x_1 f(r)F(6) & A_3^3 = -x_4 f(r)F(6) & A_4^3 = x_3 f(r)F(6)
\end{array}
\tag{3.5.31}
$$

while f(r) has to satisfy

$$r \frac{df}{dr}(r) + 4\, f(r) + gr^2(F(2)*F(6))f(r)^2 = 0 \tag{3.5.31a}$$

The solution of (3.5.31a)

$$f(r) = -\frac{2}{g}\ (F(2)*F(6))^{-1} \frac{a^2}{(r^2+a^2)r^2} \tag{3.5.32}$$

together with (3.5.31) is just the 't Hooft instanton solution with instanton number k = 1 ; this solution (3.5.31), (3.5.32) can be obtained from (3.5.24), (3.5.26) by a gauge transformation.

The equations for the static SU(2) gauge field are described by (3.5.11) and (3.5.6).

The infinitesimal symmetries for the static gauge field are obtained from those for the time-dependent case in the following way: Take a general combination of the vector fields VF(1),....,VF(18) (3.5.18), compute the prolongation components $\partial_{A^a_{\mu,4}}$ of this general vector field and impose the condition that these components are zero subject to (3.5.14), (3.5.6), [32]

$$L_V\ A^a_{\mu,4} = 0 \Big|_{\substack{(3.5.11)\\(3.5.5)}} . \tag{3.5.33}$$

A straight-forward computation then results in the Lie algebra of infinitesimal symmetries for static self dual SU(2) Yang-Mills equations

$$\begin{aligned}
VF(1) &:= (D_{A11}*DF(C(1),X1) + D_{A12}*DF(C(1),X2) + D_{A13}*DF(C(1),X3) \\
&\quad + D_{A21}*GE*A31*C(1) + D_{A22}*GE*A32*C(1) + D_{A23}*GE*A33*C(1) \\
&\quad + D_{A24}*GE*A34*C(1) - D_{A31}*GE*A21*C(1) - D_{A32}*GE*A22*C(1) \\
&\quad - D_{A33}*GE*A23*C(1) - D_{A34}*GE*A24*C(1))/GE \\
VF(2) &:= (- D_{A11}*GE*A21*C(2) - D_{A12}*GE*A22*C(2) - D_{A13}*GE*A23*C(2) \\
&\quad - D_{A14}*GE*A24*C(2) + D_{A21}*GE*A11*C(2) + D_{A22}*GE*A12*C(2) \\
&\quad + D_{A23}*GE*A13*C(2) + D_{A24}*GE*A14*C(2) - D_{A31}*DF(C(2),X1) \\
&\quad - D_{A32}*DF(C(2),X2) - D_{A33}*DF(C(2),X3))/GE \\
VF(3) &:= (D_{A11}*GE*A31*C(3) + D_{A12}*GE*A32*C(3) + D_{A13}*GE*A33*C(3) \\
&\quad + D_{A14}*GE*A34*C(3) - D_{A21}*DF(C(3),X1) - D_{A22}*DF(C(3),X2) \\
&\quad - D_{A23}*DF(C(3),X3) - D_{A31}*GE*A11*C(3) - D_{A32}*GE*A12*C(3) \\
&\quad - D_{A33}*GE*A13*C(3) - D_{A34}*GE*A14*C(3))/GE \\
VF(4) &:= D_{X1} \qquad\qquad (3.5.34) \\
VF(5) &:= D_{X2} \\
VF(6) &:= D_{X3} \\
VF(7) &:= D_{X1}*X2 - D_{X2}*X1 + D_{A11}*A12 - D_{A12}*A11 + D_{A21}*A22 - D_{A22}*A21 \\
&\quad + D_{A31}*A32 - D_{A32}*A31 \\
VF(8) &:= - D_{X1}*X3 + D_{X3}*X1 - D_{A11}*A13 + D_{A13}*A11 - D_{A21}*A23 \\
&\quad + D_{A23}*A21 - D_{A31}*A33 + D_{A33}*A31 \\
VF(9) &:= - D_{X2}*X3 + D_{X3}*X2 - D_{A12}*A13 + D_{A13}*A12 - D_{A22}*A23 \\
&\quad + D_{A23}*A22 - D_{A32}*A33 + D_{A33}*A32 \\
VF(10) &:= D_{X1}*X1 + D_{X2}*X2 + D_{X3}*X3 - D_{A11}*A11 - D_{A12}*A12 - D_{A13}*A13 \\
&\quad - D_{A14}*A14 - D_{A21}*A21 - D_{A22}*A22 - D_{A23}*A23 - D_{A24}*A24 \\
&\quad - D_{A31}*A31 - D_{A32}*A32 - D_{A33}*A33 - D_{A34}*A34
\end{aligned}$$

In (3.5.34) C(1), C(2), C(3) are arbitrary functions of $x_1,\ldots,x_3$ VF(1), VF(2), VF(3) are just the generators of the gauge transformations.
VF(7), VF(8), VF(9) generate the rotations, while VF(10) is the generator of the scale change.

In order to construct similarity solutions to the static SU(2) gauge field, we proceed in a way analogously to the one for the time-dependent field configuration.
We define the vector fields Y_1, Y_2, Y_3 by

$$\begin{aligned} Y_1 &= VF(7) - VF(2) \\ Y_2 &= VF(8) - VF(3) \\ Y_3 &= VF(9) - VF(1) \end{aligned} \qquad (3.5.35)$$

and put C(1), C(2), C(3) equal to 1.
We compute the contractions of these vector fields and the contact 1-forms (3.5.17), (3.5.6)
This results in a system of 36 equations for the functions A^a_μ
(notations as in (3.5.19), (3.5.23a))

```
1:      0 =   A21 + A12 - A11,1*X2 + A11,2*X1
2:      0 = - A11 + A22 - A21,1*X2 + A21,2*X1
3:      0 =   A32 - A31,1*X2 + A31,2*X1
4:      0 = - A11 + A22 - A12,1*X2 + A12,2*X1
5:      0 = - A21 - A12 - A22,1*X2 + A22,2*X1
6:      0 = - A31 - A32,1*X2 + A32,2*X1
7:      0 =   A23 - A13,1*X2 + A13,2*X1
8:      0 = - A13 - A23,1*X2 + A23,2*X1
9:      0 = - A33,1*X2 + A33,2*X1
10:     0 =  A24 - A14,1*X2 + A14,2*X1
11:     0 = - A14 - A24,1*X2 + A24,2*X1
12:     0 = - A34,1*X2 + A34,2*X1
13:     0 = - A31 - A13 + A11,1*X3 - A11,3*X1
14:     0 = - A23 + A21,1*X3 - A21,3*X1
15:     0 =   A11 - A33 + A31,1*X3 - A31,3*X1
16:     0 = - A32 + A12,1*X3 - A12,3*X1
17:     0 =   A22,1*X3 - A22,3*X1
18:     0 =   A12 + A32,1*X3 - A32,3*X1                          (3.5.36)
19:     0 =   A11 - A33 + A13,1*X3 - A13,3*X1
20:     0 =   A21 + A23,1*X3 - A23,3*X1
21:     0 =   A31 + A13 + A33,1*X3 - A33,3*X1
22:     0 = - A34 + A14,1*X3 - A14,3*X1
23:     0 =   A24,1*X3 - A24,3*X1
24:     0 =   A14 + A34,1*X3 - A34,3*X1
25:     0 =   A11,2*X3 - A11,3*X2
26:     0 = - A31 + A21,2*X3 - A21,3*X2
27:     0 =   A21 + A31,2*X3 - A31,3*X2
28:     0 = - A13 + A12,2*X3 - A12,3*X2
29:     0 = - A32 - A23 + A22,2*X3 - A22,3*X2
30:     0 =   A22 - A33 + A32,2*X3 - A32,3*X2
31:     0 =   A12 + A13,2*X3 - A13,3*X2
32:     0 =   A22 - A33 + A23,2*X3 - A23,3*X2
33:     0 =   A32 + A23 + A33,2*X3 - A33,3*X2
34:     0 =   A14,2*X3 - A14,3*X2
35:     0 = - A34 + A24,2*X3 - A24,3*X2
36:     0 =   A24 + A34,2*X3 - A34,3*X2
```

We shall now indicate in some detail how to solve (3.5.36).

Note that, due to (3.5.36(12))

$$A_4^3 = f^1(r_{12}, z) \qquad (3.5.37)$$

where

$$r_{12} = \sqrt{x_1^2 + x_2^2} \; ;$$

and due to (3.5.36(24))

$$A_4^1 = x_1\{\partial_2 f^1(r_{12},x_3) - \frac{x_3}{r_{12}}\, \partial_1 f^1(r_{12},x_3)\}, \qquad (3.5.38)$$

where ∂_i denotes differentiation with respect to the i-th component.
Now let

$$\partial_2 f^1(r_{12},x_3) - \frac{x_3}{r_{12}}\, \partial_1 f^1(r_{12},x_3) = h(r_{12},x_3). \qquad (3.5.39)$$

Substitution of (3.5.37), (3.5.39) into (3.5.36.(22)) results in

$$f^1(r_{12},x_3) = x_3 h_1(r_{12},x_3) + \frac{x_1^2}{r_{xy}}\, \partial_1 h(r_{12},x_3) - x_1^2\, \partial_2 h(r_{12},x_3)$$

or

$$f^1(r_{12},x_3) - x_3 h_1(r_{12},x_3) = x_1^2\, (\frac{1}{r_{12}}\, \partial_1 h(r_{12},x_3) - \partial_2 h(r_{12},x_3)) \qquad (3.5.40)$$

Differentiation of (3.5.40) with respect to x_1, x_2 yields

$$\frac{1}{r_{12}}\, \partial_1 h(r_{12},x_3) - \partial_2 h(r_{12},x_3) = 0 \qquad (3.5.41a)$$

$$f^1(r_{12},x_3) = x_3 h_1(r_{12},x_3) \qquad (3.5.41b)$$

From (3.5.41b) we obtain

$$h(r_{12},x_3) = 1(r) \qquad (3.5.42)$$

where

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \qquad (3.5.41a)$$

and finally due to (3.5.41b), (3.5.38) and (3.5.36(36))

$$A_4^1 = x_1 l(r) \;;\; A_4^2 = x_2 l(r) \;;\; A_4^3 = x_3 l(r) \tag{3.5.43}$$

Handling the remaining system in a similar way, a straightforward but tedious computation leads to the general solution of (3.5.36) i.e.,

$$\begin{array}{lll}
A_1^1 = \frac{1}{2} x_1^2 f(r) + k(r) & A_2^1 = \frac{1}{2} x_1 x_2 f(r) - x_3 u(r) & A_3^1 = \frac{1}{2} x_1 x_3 f(r) + x_2 u(r) \\
A_1^2 = \frac{1}{2} x_1 x_2 f(r) + x_3 u(r) & A_2^2 = \frac{1}{2} x_2^2 f(r) + k(r) & A_3^2 = \frac{1}{2} x_2 x_3 f(r) - x_1 u(r) \\
A_1^3 = \frac{1}{2} x_1 x_3 f(r) - x_2 u(r) & A_2^3 = \frac{1}{2} x_2 x_3 f(r) + x_1 u(r) & A_3^3 = \frac{1}{2} x_3^2 f(r) + k(r) \\
A_4^1 = x_1 l(r) & A_4^2 = x_2 l(r) & A_4^3 = x_3 l(r)
\end{array} \tag{3.5.44}$$

where u, l, k, f are functions of r.

Substitution of (3.5.44) into (3.5.14), (3.5.6) yields a system of three ordinary differential equations for the functions u, l, k, f

$$\begin{aligned}
&\frac{dl(r)}{dr} + \frac{du(r)}{dr} - gru(r)^2 - gru(r)l(r) + \frac{1}{2} grf(r)k(r) = 0 \\
&r^2 \frac{du(r)}{dr} + 2ru(r) - rl(r) - gr^3 u(r)l(r) + grk(r)^2 + \frac{1}{2} gr^3 f(r)k(r) = 0 \\
&\frac{dk(r)}{dr} - \frac{1}{2} rf(r) - grk(r)u(r) - grl(r)k(r) - \frac{1}{2} gr^3 f(r)u(r) = 0
\end{aligned} \tag{3.5.45}$$

If we choose

$$f(r) \equiv 0, \quad k(r) \equiv 0, \quad l(r) = \frac{h(r)}{r}, \quad u(r) = -\frac{a(r)}{r} \tag{3.5.46}$$

we are led by (3.5.46) (3.5.44) to the monopole solution, obtained by Prasad & Sommerfield [31] by imposing the ansatz (3.5.44), (3.5.46).

Chapter 4

LIE-BACKLUND TRANSFORMATIONS

4.0 Introduction

In this chapter some applications of the software, described in chapter 2, to the computation of Lie-Bäcklund transformations of differential equations shall be given. For an introduction to Lie-Bäcklund transformations we refer to chapter 1.

In section 1 we describe the solution method in some detail for the computation of Lie-Bäcklund transformations of Burgers' equation, leading to results which are in agreement with those of Vinogradov [39], where the complete algebra is given.

In section 2 we study Lie-Bäcklund transformations of the Classical Boussinesq equation. Recently Ito [14] discovered a generating operator, leading to an infinite hierarchy of commuting Lie-Bäcklund transformations. By the method described in chapter 1 we derive a Lie-Bäcklund transformation whose action on the other Lie-Bäcklund transformations is similar to the action of the generating operator found by Ito.

In section 3 we construct second and third order Lie-Bäcklund transformations of the Massive Thirring Model [15].

In section 4 we introduce the notion of nonlocal Lie-Bäcklund transformations. We derive conserved vectors useful in the study for nonlocal Lie-Bäcklund transformations and we derive two nonlocal Lie-Bäcklund transformations probably generating an infinite hierarchy of Lie-Bäcklund transformations of the Massive Thirring Model. The Lie algebra structure of the Lie-Bäcklund transformations is given.

4.1 Lie-Bäcklund transformations of Burgers' equation

To demonstrate the technique of computing Lie-Bäcklund transformations we shall discuss the relatively simple case of Burgers' equation. This equation is given by

$$u_t = uu_x + u_{xx}. \tag{4.1.1}$$

Since we shall be working in the local jet bundle formulation we have to take into account the differential consequences of (4.1.1) i.e.,

$$\begin{aligned} u_{xxx} &= u_{xt} - uu_{xx} - u_x^2 \\ u_{xxt} &= u_{tt} - uu_{xt} - u_x u_t \\ &\vdots \end{aligned} \tag{4.1.2}$$

We can construct an ideal I generated by differential 1-forms associated to (4.1.1) in $\mathcal{Y}$(1.1.15), where local coordinates are given by

$$\{(X(1),\cdots,X(7))\} = \{(x,t,u,u_x,u_t,u_{xt},u_{tt})\}, \tag{4.1.3}$$

defined by

$$\begin{aligned} \alpha(1) &= du - u_x dx - u_t dt \\ \alpha(2) &= du_x - (u_t - uu_x)dx - u_{xt}dt \\ \alpha(3) &= du_t - u_{xt}dx - u_{tt}dt. \end{aligned} \tag{4.1.4}$$

We now define the total derivative vector fields defined on

$$\mathcal{Y}^\infty = \{(x,t,u,u_x,u_t,u_{xt},u_{tt},u_{xtt},u_{ttt},\cdots)\} \tag{4.1.5}$$

by (1.2.5.)

$$D_x = \partial_x + u_x\partial_u + u_{xt}\partial_{u_t} + \cdots \tag{4.1.6}$$

$$D_t = \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + \cdots$$

We define the 3 times prolonged ideal D^3I of I (4.1.4) generated by the differential 1-forms

$$
\begin{aligned}
\alpha(1) &= du - u_{01}dt - u_1dx \\
\alpha(2) &= du_1 - u_{11}dt - (u_{01} - u\,u_1)dx \\
\alpha(3) &= du_{01} - u_{02}\,dt - u_{11}dx \\
\alpha(4) &= du_{11} - u_{12}dt - (u_{02} - u_{01}u_1 - u\,u_{11})dx \\
\alpha(5) &= du_{02} - u_{03}dt - u_{12}dx \\
\alpha(6) &= du_{12} - u_{13}dt - (u_{03} - u\,u_{12} - 2u_{01}u_{11} - u_{02}u_1)dx \\
\alpha(7) &= du_{03} - u_{04}dt - u_{13}dx \\
\alpha(8) &= du_{13} - u_{14}dt - (u_{04} - u\,u_{13} - 3u_{01}u_{12} - 3u_{02}u_{11} - u_{03}u_1)dx \\
\alpha(9) &= du_{04} - u_{05}dt - u_{14}dx,
\end{aligned}
\qquad (4.1.7)
$$

where we introduced

$$u_{ij} = u_{x\ldots xt\ldots t} \quad (u_{i0} \equiv u_i\ ,\ u_{00} \equiv u). \qquad (4.1.8)$$

Since the vector fields fD_x, gD_t, where f,g are arbitrarily, smooth functions of a finite number of variables, satisfy the Lie-Bäcklund condition

$$L_V(D^\infty I) \subset D^\infty I \qquad (4.1.9)$$

in an obvious way, (chapter 1) our search for Lie-Bäcklund transformations shall be restricted to the search for vertical vector fields (1.2.15)

$$V = \eta\partial_u + \cdots \qquad (4.1.10)$$

We shall construct third order Lie-Bäcklund transformations of (4.1.1) by requiring

$$\eta = \eta(x,t,u,\ldots,u_{12},u_{03}). \qquad (4.1.11)$$

V, defined by (4.1.10),(4.1.11) is a Lie-Bäcklund transformations of (4.1.1) if

$$L_V I \subset D^3 I. \tag{4.1.12}$$

Now let V be defined by

$$V = \sum_{i=3}^{7} F(i)\partial_{X(i)} \tag{4.1.13}$$

then using the symbolic procedure INFSYM(*,*,*) described in chapter 2 we derive from (4.1.12) an overdetermined system of 12 partial differential equations for the functions F(i) (i=3,...7) (Appendix <u>BURGERS</u> 1). The intermediate computer results, referred to by Appendix <u>BURGERS</u> * are enclosed at the end of this section.

In fact we are only interested in the ∂_u-component of the vector field V i.e., F(3), since the other components can be obtained from F(3) by total partial differentiation

$$V = \eta\partial_u + (D_x\eta)\partial_{u_x} + (D_t\eta)\partial_{u_t} + \cdots \tag{4.1.14}$$

and which is in agreement with VER(3),VER(4),VER(5),VER(9) of Appendix <u>BURGERS</u> 1.

The solution procedure of this overdetermined system is analogous to the method for the search for ordinary symmetries in chapter 3.

Using the integration package we first solve the equations VER(1), VER(2). Then we solve VER(3) for F(5), VER(4) for F(4), VER(5) for F(6) and VER(9) for F(7).

Substitution of the results into VER(8), VER(10) then results in

F(3):= F(17) (4.1.15)

F(17) depends on X(1),...,X(9). (4.1.15a)

VER(6) breaks up into three new equations VER(13), VER(14), VER(15). (Appendix <u>BURGERS</u> 2)

From VER(13) we conclude that F(17) (F(3) : (4.1.15)) is linear with respect to X(9) and due to VER(14) we arrive at

F(3):= F(19) + x(9)*F(26), (4.1.16)

where in (4.1.16)

F(26) depends on X(2),
F(19) depends on X(1),...,X(8). (4.1.16a)

VER(15) breaks up into VER(16), VER(17), VER(18).
(Appendix <u>BURGERS</u> 3)

From VER(16) we see that F(19) is linear with respect to X(8) and combination with VER(17) yields

$$F(3):= F(28) + X(8)*((\tfrac{1}{2})*DF(F(26),X(2))*x(1) + F(34)) + X(9)*F(26) \qquad (4.1.17)$$

whereas

F(28) depends on X(1),...,X(7)
F(34) depends on X(2). (4.1.17a)

Finally VER(18) leads to the three remaining equations VER(19), VER(20), VER(21).
(Appendix <u>BURGERS</u> 4).

Proceeding in this way the computation of the general solution of (4.1.12-14) is straight-forward, resulting in 27 independent vector fields.
Due to the fact that (4.1.1) is an evolution equation, transformation of the results into a standard form, where

$$\begin{aligned}
&X(1) = x,\ X(2) = t,\ X(3) = u,\ X(4) = u_x = u_1,\ X(5) = u_t = u_2 + u_1 u\\
&X(6) = u_{xt} = u_3 + u_2 u + u_1^2\\
&X(7) = u_{tt} = u_4 + 2u_3 u + 4u_2 u_1 + u_2 u^2 + 2u_1^2 u \qquad (4.1.18a)\\
&X(8) = u_{xtt} = u_5 + 2u_4 u + 6u_3 u_1 + u_3 u^2 + 4u_2^2 + 6u_2 u_1 u + 2u_1^3\\
&X(9) = u_{ttt} = u_6 + 3u_5 u + 9u_4 u_1 + 3u_4 u^2 + 16u_3 u_2 + 18u_3 u_1 u\\
&\qquad + u_3 u^3 + 12u_2^2 u + 18u_2 u_1^2 + 9u_2 u_1 u^2 + 6\, u_1^3 u
\end{aligned}$$

leads to the ∂_u-components of the Lie-Bäcklund transformations

$$GG(1) := U_2 + U_1 \ast U$$

$$GG(2) := U_1$$

$$GG(3) := T \ast U_1 + 1$$

$$GG(4) := X \ast U_1 + 2 \ast T \ast (U_2 + U_1 \ast U) + U \tag{4.1.18b}$$

$$GG(5) := X \ast T \ast U_1 + X + T^2 \ast (U_2 + U_1 \ast U) + T \ast U$$

$$GG(6) := 2 \ast X \ast (U_2 + U_1 \ast U) + T \ast (4 \ast U_3 + 6 \ast U_2 \ast U + 6 \ast U_1^2 + 3 \ast U_1 \ast U^2) + U^2$$

$$GG(7) := X^2 \ast U_1 + 4 \ast X \ast T \ast (U_2 + U_1 \ast U) + 2 \ast X \ast U + T^2 \ast (4 \ast U_3 + 6 \ast U_2 \ast U + 6 \ast U_1^2 + 3 \ast U_1 \ast U^2) + 2 \ast T \ast U^2 - 6$$

$$GG(8) := 3 \ast X^2 \ast T \ast U_1 + 3 \ast X^2 + 6 \ast X \ast T^2 \ast (U_2 + U_1 \ast U) + 6 \ast X \ast T \ast U + T^3 \ast (4 \ast U_3 + 6 \ast U_2 \ast U + 6 \ast U_1^2 + 3 \ast U_1 \ast U^2) + 3 \ast T^2 \ast (4 \ast U + U^2) + 6 \ast T$$

$$GG(9) := 4 \ast U_3 + 6 \ast U_2 \ast U + 6 \ast U_1^2 + 3 \ast U_1 \ast U^2$$

while GG(10),...,GG(20) are given in Appendix BURGERS 5.
The explicit results for the ∂_u-components GG(21),...,GG(27) are in [18].

The Lie-Bäcklund transformations defined by GG(1),...,GG(5) are equivalent to classical symmetries; GG(9), GG(14), GG(20) were found by Olver [28]; GG(6) by ten Eikelder [7], while the complete result is in agreement with the result obtained by Vinogradov [39], where the complete algebra is given.

Appendix BURGERS 1

```
LISP DEPL!*_'(((F 7) (X 1) (X 2) (X 3) (X 4) (X 5) (X
6) (X 7) (X 8) (X 9) (X 10) (X 11) (X 12) (X 13)) ((F 6) (X 1) (X
 2) (X 3) (X 4) (X 5) (X 6) (X 7) (X 8) (X 9) (X 10) (X 11) (X 12
) (X 13)) ((F 5) (X 1) (X 2) (X 3) (X 4) (X 5) (X 6) (X 7) (X 8)
(X 9) (X 10) (X 11) (X 12) (X 13)) ((F 4) (X 1) (X 2) (X 3) (X 4)
 (X 5) (X 6) (X 7) (X 8) (X 9) (X 10) (X 11) (X 12) (X 13)) ((F 3
) (X 1) (X 2) (X 3) (X 4) (X 5) (X 6) (X 7) (X 8) (X 9) (X 10) (X
 11) (X 12) (X 13)));

VER(1) := DF(F(3),X(13))$

VER(2) := DF(F(3),X(12))$

VER(3) := X(13)*DF(F(3),X(11)) + X(12)*DF(F(3),X(10)) + DF(F(3),X
(3))*X(5) + DF(F(3),X(4))*X(6) + DF(F(3),X(5))*X(7) + DF(F(3),X(6
))*X(8) + DF(F(3),X(7))*X(9) + DF(F(3),X(8))*X(10) + DF(F(3),X(9)
)*X(11) + DF(F(3),X(2)) - F(5)$

VER(4) := X(12)*DF(F(3),X(11)) + DF(F(3),X(3))*X(4) + X(5)*DF(F(3
),X(4)) - 2*X(5)*X(6)*DF(F(3),X(8)) - X(5)*DF(F(3),X(6))*X(4) - 3
*X(5)*X(8)*DF(F(3),X(10)) - DF(F(3),X(4))*X(4)*X(3) + X(6)*DF(F(3
),X(5)) - 3*X(6)*X(7)*DF(F(3),X(10)) - X(6)*DF(F(3),X(6))*X(3) +
X(7)*DF(F(3),X(6)) - X(7)*DF(F(3),X(8))*X(4) + X(8)*DF(F(3),X(7))
 - X(8)*DF(F(3),X(8))*X(3) + X(9)*DF(F(3),X(8)) - X(9)*DF(F(3),X(
10))*X(4) + X(10)*DF(F(3),X(9)) - X(10)*DF(F(3),X(10))*X(3) + X(
11)*DF(F(3),X(10)) + DF(F(3),X(1)) - F(4)$

VER(5) := X(13)*DF(F(4),X(11)) + X(12)*DF(F(4),X(10)) + X(5)*DF(F
(4),X(3)) + X(6)*DF(F(4),X(4)) + X(7)*DF(F(4),X(5)) + X(8)*DF(F(4
),X(6)) + X(9)*DF(F(4),X(7)) + X(10)*DF(F(4),X(8)) + X(11)*DF(F(4
),X(9)) + DF(F(4),X(2)) - F(6)$
```

Appendix BURGERS 1

```
VER(6) := X(12)*DF(F(4),X(11)) - 2*X(5)*X(6)*DF(F(4),X(8)) - 3*X(
5)*X(8)*DF(F(4),X(10)) + X(5)*DF(F(4),X(4)) - X(5)*DF(F(4),X(6))*
X(4) - 3*X(6)*X(7)*DF(F(4),X(10)) + X(6)*DF(F(4),X(5)) - X(6)*DF(
F(4),X(6))*X(3) + X(7)*DF(F(4),X(6)) - X(7)*DF(F(4),X(8))*X(4) +
X(8)*DF(F(4),X(7)) - X(8)*DF(F(4),X(8))*X(3) + X(9)*DF(F(4),X(8))
 - X(9)*DF(F(4),X(10))*X(4) + X(10)*DF(F(4),X(9)) - X(10)*DF(F(4)
,X(10))*X(3) + X(11)*DF(F(4),X(10)) + F(4)*X(3) + DF(F(4),X(3))*X
(4) - DF(F(4),X(4))*X(4)*X(3) + DF(F(4),X(1)) + X(4)*F(3) - F(5)$

VER(7) := DF(F(4),X(13))$

VER(8) := DF(F(4),X(12))$

VER(9) := X(13)*DF(F(5),X(11)) + X(12)*DF(F(5),X(10)) + X(5)*DF(F
(5),X(3)) + X(6)*DF(F(5),X(4)) + X(7)*DF(F(5),X(5)) + X(8)*DF(F(5
),X(6)) + X(9)*DF(F(5),X(7)) + X(10)*DF(F(5),X(8)) + X(11)*DF(F(5
),X(9)) + DF(F(5),X(2)) - F(7)$

VER(10) := X(12)*DF(F(5),X(11)) - 2*X(5)*X(6)*DF(F(5),X(8)) - 3*X
(5)*X(8)*DF(F(5),X(10)) + X(5)*DF(F(5),X(4)) - X(5)*DF(F(5),X(6))
*X(4) - 3*X(6)*X(7)*DF(F(5),X(10)) + X(6)*DF(F(5),X(5)) - X(6)*DF
(F(5),X(6))*X(3) + X(7)*DF(F(5),X(6)) - X(7)*DF(F(5),X(8))*X(4)
 + X(8)*DF(F(5),X(7)) - X(8)*DF(F(5),X(8))*X(3) + X(9)*DF(F(5),X(
8)) - X(9)*DF(F(5),X(10))*X(4) + X(10)*DF(F(5),X(9)) - X(10)*DF(F
(5),X(10))*X(3) + X(11)*DF(F(5),X(10)) + DF(F(5),X(3))*X(4) - DF(
F(5),X(4))*X(4)*X(3) + DF(F(5),X(1)) - F(6)$

VER(11) := DF(F(5),X(13))$

VER(12) := DF(F(5),X(12))$
```

Appendix <u>BURGERS</u> 2

```
LISP DEPL!*_'
(((F 17) (X 1) (X 2) (X 3) (X 4) (X 5) (X 6) (X 7) (X 8) (X 9)) (
REST (X 1) (X 2) (X 3) (X 4) (X 5) (X 6) (X 7) (X 8) (X 9) (X 10)
 (X 11) (X 12) (X 13));
F(3) := F(17)$

VER(13) := DF(F(17),X(9),2)$

VER(14) := 2*(DF(F(17),X(9),X(3))*X(4) - 2*X(5)*X(6)*DF(F(17),X(9
),X(8)) + X(5)*DF(F(17),X(9),X(4)) - X(5)*DF(F(17),
X(9),X(6))*X(4) + X(6)*DF(F(17),X(9),X(5)) - X(6)*
DF(F(17),X(9),X(6))*X(3) + X(7)*DF(F(17),X(9),X(6))
 - X(7)*DF(F(17),X(9),X(8))*X(4) + X(8)*DF(F(17),X(
9),X(7)) - X(8)*DF(F(17),X(9),X(8))*X(3) + X(9)*DF(
F(17),X(9),X(8)) - DF(F(17),X(9),X(4))*X(4)*X(3) +
DF(F(17),X(9),X(1)))$

VER(15) := 2*DF(F(17),X(7),X(1))*X(8) + F(17)*X(4) - DF(F(17),X(4
))*X(4)**2 - DF(F(17),X(5))*X(5)*X(4) - DF(F(17),X(6))*X(5)**2 +
DF(F(17),X(6))*X(5)*X(4)*X(3) - 2*DF(F(17),X(6))*X(6)*X(4) - 2*DF
(F(17),X(7))*X(5)*X(6) - DF(F(17),X(7))*X(7)*X(4) + 2*DF(F(17),X(
8))*X(5)**2*X(4) + 2*DF(F(17),X(8))*X(5)*X(6)*X(3) - 3*DF(F(17),X
(8))*X(5)*X(7) - 2*DF(F(17),X(8))*X(6)**2 + DF(F(17),X(8))*X(7)*X
(4)*X(3) - 2*DF(F(17),X(8))*X(8)*X(4) - 3*DF(F(17),X(9))*X(5)*X(8
) - 3*DF(F(17),X(9))*X(6)*X(7) - DF(F(17),X(9))*X(9)*X(4) - DF(F(
17),X(2)) + DF(F(17),X(3),2)*X(4)**2 + 2*DF(F(17),X(4),X(3))*X(5)
*X(4) - 2*DF(F(17),X(4),X(3))*X(4)**2*X(3) + 2*DF(F(17),X(5),X(3)
)*X(6)*X(4) - 2*DF(F(17),X(6),X(3))*X(5)*X(4)**2 - 2*DF(F(17),X(6
),X(3))*X(6)*X(4)*X(3) + 2*DF(F(17),X(6),X(3))*X(7)*X(4) + 2*DF(F
(17),X(7),X(3))*X(8)*X(4) - 4*DF(F(17),X(8),X(3))*X(5)*X(6)*X(4)
 - 2*DF(F(17),X(8),X(3))*X(7)*X(4)**2 - 2*DF(F(17),X(8),X(3))*X(8
)*X(4)*X(3) + 2*DF(F(17),X(8),X(3))*X(9)*X(4) + 4*X(5)**2*X(6)**2
*DF(F(17),X(8),2) - 4*X(5)**2*X(6)*DF(F(17),X(8),X(4)) + 4*X(5)**
2*X(6)*DF(F(17),X(8),X(6))*X(4) + X(5)**2*DF(F(17),X(4),2) - 2*X(
5)**2*DF(F(17),X(6),X(4))*X(4) + X(5)**2*DF(F(17),X(6),2)*X(4)**2
```

Appendix BURGERS 2

```
 - 4*X(5)*X(6)**2*DF(F(17),X(8),X(5)) + 4*X(5)*X(6)**2*DF(F(17),X
(8),X(6))*X(3) - 4*X(5)*X(6)*X(7)*DF(F(17),X(8),X(6)) + 4*X(5)*X(
6)*X(7)*DF(F(17),X(8),2)*X(4) - 4*X(5)*X(6)*X(8)*DF(F(17),X(8),X(
7)) + 4*X(5)*X(6)*X(8)*DF(F(17),X(8),2)*X(3) - 4*X(5)*X(6)*X(9)*
DF(F(17),X(8),2) + 2*X(5)*X(6)*DF(F(17),X(5),X(4)) - 2*X(5)*X(6)*
DF(F(17),X(6),X(4))*X(3) + 4*X(5)*X(6)*DF(F(17),X(8),X(4))*X(4)*X
(3) - 2*X(5)*X(6)*DF(F(17),X(6),X(5))*X(4) + 2*X(5)*X(6)*DF(F(17)
,X(6),2)*X(4)*X(3) - 4*X(5)*X(6)*DF(F(17),X(8),X(1)) + 2*X(5)*X(7
)*DF(F(17),X(6),X(4)) - 2*X(5)*X(7)*DF(F(17),X(8),X(4))*X(4) - 2*
X(5)*X(7)*DF(F(17),X(6),2)*X(4) + 2*X(5)*X(7)*DF(F(17),X(8),X(6))
*X(4)**2 + 2*X(5)*X(8)*DF(F(17),X(7),X(4)) - 2*X(5)*X(8)*DF(F(17)
,X(8),X(4))*X(3) - 2*X(5)*X(8)*DF(F(17),X(7),X(6))*X(4) + 2*X(5)*
X(8)*DF(F(17),X(8),X(6))*X(4)*X(3) + 2*X(5)*X(9)*DF(F(17),X(8),X(
4)) - 2*X(5)*X(9)*DF(F(17),X(8),X(6))*X(4) - 2*X(5)*DF(F(17),X(4)
,2)*X(4)*X(3) + 2*X(5)*DF(F(17),X(6),X(4))*X(4)**2*X(3) + 2*X(5)*
DF(F(17),X(4),X(1)) - 2*X(5)*DF(F(17),X(6),X(1))*X(4) + X(6)**2*
DF(F(17),X(5),2) - 2*X(6)**2*DF(F(17),X(6),X(5))*X(3) + X(6)**2*
DF(F(17),X(6),2)*X(3)**2 + 2*X(6)*X(7)*DF(F(17),X(6),X(5)) - 2*X(
6)*X(7)*DF(F(17),X(8),X(5))*X(4) - 2*X(6)*X(7)*DF(F(17),X(6),2)*X
(3) + 2*X(6)*X(7)*DF(F(17),X(8),X(6))*X(4)*X(3) + 2*X(6)*X(8)*DF(
F(17),X(7),X(5)) - 2*X(6)*X(8)*DF(F(17),X(8),X(5))*X(3) - 2*X(6)*
X(8)*DF(F(17),X(7),X(6))*X(3) + 2*X(6)*X(8)*DF(F(17),X(8),X(6))*X
(3)**2 + 2*X(6)*X(9)*DF(F(17),X(8),X(5)) - 2*X(6)*X(9)*DF(F(17),X
(8),X(6))*X(3) - 2*X(6)*DF(F(17),X(5),X(4))*X(4)*X(3) + 2*X(6)*DF
(F(17),X(6),X(4))*X(4)*X(3)**2 - 2*X(6)*DF(F(17),X(6),X(1))*X(3)
 + 2*X(6)*DF(F(17),X(5),X(1)) + X(7)**2*DF(F(17),X(6),2) - 2*X(7)
**2*DF(F(17),X(8),X(6))*X(4) + X(7)**2*DF(F(17),X(8),2)*X(4)**2
 + 2*X(7)*X(8)*DF(F(17),X(7),X(6)) - 2*X(7)*X(8)*DF(F(17),X(8),X(
6))*X(3) - 2*X(7)*X(8)*DF(F(17),X(8),X(7))*X(4) + 2*X(7)*X(8)*DF(
F(17),X(8),2)*X(4)*X(3) + 2*X(7)*X(9)*DF(F(17),X(8),X(6)) - 2*X(7
)*X(9)*DF(F(17),X(8),2)*X(4) - 2*X(7)*DF(F(17),X(6),X(4))*X(4)*X(
3) + 2*X(7)*DF(F(17),X(8),X(4))*X(4)**2*X(3) - 2*X(7)*DF(F(17),X(
8),X(1))*X(4) + 2*X(7)*DF(F(17),X(6),X(1)) + X(8)**2*DF(F(17),X(7
),2) - 2*X(8)**2*DF(F(17),X(8),X(7))*X(3) + X(8)**2*DF(F(17),X(8)
,2)*X(3)**2 + 2*X(8)*X(9)*DF(F(17),X(8),X(7)) - 2*X(8)*X(9)*DF(F(
```

Appendix BURGERS 2

```
17),X(8),2)*X(3) - 2*X(8)*DF(F(17),X(7),X(4))*X(4)*X(3) + 2*X(8)*
DF(F(17),X(8),X(4))*X(4)*X(3)**2 - 2*X(8)*DF(F(17),X(8),X(1))*X(3
) + X(9)**2*DF(F(17),X(8),2) - 2*X(9)*DF(F(17),X(8),X(4))*X(4)*X(
3) + 2*X(9)*DF(F(17),X(8),X(1)) + DF(F(17),X(4),2)*X(4)**2*X(3)**
2 + 2*DF(F(17),X(3),X(1))*X(4) + DF(F(17),X(1),2) + DF(F(17),X(1)
)*X(3) - 2*DF(F(17),X(4),X(1))*X(4)*X(3)$
```

Appendix BURGERS 3

```
LISP DEPL!*_'
(((F 26) (X 2)) ((F 19) (X 1) (X 2) (X 3) (X 4) (X 5) (X 6) (X 7)
 (X 8)) (REST (X 1) (X 2) (X 3) (X 4) (X 5) (X 6) (X 7) (X 8) (X
9) (X 10) (X 11) (X 12) (X 13)));
F(3) := F(19) + X(9)*F(26)$

VER(16) := DF(F(19),X(8),2)$

VER(17) :=  - 4*X(5)*X(6)*DF(F(19),X(8),2) - 2*X(5)*DF(F(19),X(8)
,X(6))*X(4) + 2*X(5)*DF(F(19),X(8),X(4)) - 2*X(6)*DF(F(19),X(8),X
(6))*X(3) + 2*X(6)*DF(F(19),X(8),X(5)) + 2*X(7)*DF(F(19),X(8),X(6
)) - 2*X(7)*DF(F(19),X(8),2)*X(4) - 2*X(8)*DF(F(19),X(8),2)*X(3)
 + 2*X(8)*DF(F(19),X(8),X(7)) - DF(F(26),X(2)) + 2*DF(F(19),X(8),
X(3))*X(4) - 2*DF(F(19),X(8),X(4))*X(4)*X(3) + 2*DF(F(19),X(8),X(
1))$

VER(18) := 2*DF(F(19),X(3),X(1))*X(4) + DF(F(19),X(1))*X(3) + DF(
F(19),X(1),2) + F(19)*X(4) + 4*X(5)**2*X(6)**2*DF(F(19),X(8),2)
 + 4*X(5)**2*X(6)*DF(F(19),X(8),X(6))*X(4) - 4*X(5)**2*X(6)*DF(F(
19),X(8),X(4)) - X(5)**2*DF(F(19),X(6)) + 2*X(5)**2*DF(F(19),X(8)
)*X(4) - 2*X(5)**2*DF(F(19),X(6),X(4))*X(4) + X(5)**2*DF(F(19),X(
6),2)*X(4)**2 + X(5)**2*DF(F(19),X(4),2) + 4*X(5)*X(6)**2*DF(F(19
),X(8),X(6))*X(3) - 4*X(5)*X(6)**2*DF(F(19),X(8),X(5)) - 4*X(5)*X
(6)*X(7)*DF(F(19),X(8),X(6)) + 4*X(5)*X(6)*X(7)*DF(F(19),X(8),2)*
X(4) + 4*X(5)*X(6)*X(8)*DF(F(19),X(8),2)*X(3) - 4*X(5)*X(6)*X(8)*
DF(F(19),X(8),X(7)) - 2*X(5)*X(6)*DF(F(19),X(7)) + 2*X(5)*X(6)*DF
(F(19),X(8))*X(3) - 4*X(5)*X(6)*DF(F(19),X(8),X(3))*X(4) + 2*X(5)
*X(6)*DF(F(19),X(5),X(4)) - 2*X(5)*X(6)*DF(F(19),X(6),X(4))*X(3)
 + 4*X(5)*X(6)*DF(F(19),X(8),X(4))*X(4)*X(3) - 2*X(5)*X(6)*DF(F(
19),X(6),X(5))*X(4) + 2*X(5)*X(6)*DF(F(19),X(6),2)*X(4)*X(3) - 4*
X(5)*X(6)*DF(F(19),X(8),X(1)) - 3*X(5)*X(7)*DF(F(19),X(8)) + 2*X(
5)*X(7)*DF(F(19),X(8),X(6))*X(4)**2 + 2*X(5)*X(7)*DF(F(19),X(6),X
(4)) - 2*X(5)*X(7)*DF(F(19),X(8),X(4))*X(4) - 2*X(5)*X(7)*DF(F(19
),X(6),2)*X(4) - 3*X(5)*X(8)*F(26) + 2*X(5)*X(8)*DF(F(19),X(8),X(
6))*X(4)*X(3) - 2*X(5)*X(8)*DF(F(19),X(8),X(4))*X(3) + 2*X(5)*X(8
```

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```
)*DF(F(19),X(7),X(4)) - 2*X(5)*X(8)*DF(F(19),X(7),X(6))*X(4) - X(
5)*DF(F(19),X(5))*X(4) + X(5)*DF(F(19),X(6))*X(4)*X(3) + 2*X(5)*
DF(F(19),X(4),X(3))*X(4) - 2*X(5)*DF(F(19),X(6),X(3))*X(4)**2 + 2
*X(5)*DF(F(19),X(6),X(4))*X(4)**2*X(3) - 2*X(5)*DF(F(19),X(4),2)*
X(4)*X(3) + 2*X(5)*DF(F(19),X(4),X(1)) - 2*X(5)*DF(F(19),X(6),X(1
))*X(4) - 2*X(6)**2*DF(F(19),X(8)) - 2*X(6)**2*DF(F(19),X(6),X(5)
)*X(3) + X(6)**2*DF(F(19),X(6),2)*X(3)**2 + X(6)**2*DF(F(19),X(5)
,2) - 3*X(6)*X(7)*F(26) + 2*X(6)*X(7)*DF(F(19),X(8),X(6))*X(4)*X(
3) + 2*X(6)*X(7)*DF(F(19),X(6),X(5)) - 2*X(6)*X(7)*DF(F(19),X(6),
2)*X(3) - 2*X(6)*X(7)*DF(F(19),X(8),X(5))*X(4) + 2*X(6)*X(8)*DF(F
(19),X(8),X(6))*X(3)**2 - 2*X(6)*X(8)*DF(F(19),X(8),X(5))*X(3) -
2*X(6)*X(8)*DF(F(19),X(7),X(6))*X(3) + 2*X(6)*X(8)*DF(F(19),X(7),
X(5)) - 2*X(6)*DF(F(19),X(6))*X(4) + 2*X(6)*DF(F(19),X(5),X(3))*X
(4) - 2*X(6)*DF(F(19),X(6),X(3))*X(4)*X(3) - 2*X(6)*DF(F(19),X(5)
,X(4))*X(4)*X(3) + 2*X(6)*DF(F(19),X(6),X(4))*X(4)*X(3)**2 - 2*X(
6)*DF(F(19),X(6),X(1))*X(3) + 2*X(6)*DF(F(19),X(5),X(1)) - 2*X(7)
**2*DF(F(19),X(8),X(6))*X(4) + X(7)**2*DF(F(19),X(8),2)*X(4)**2
 + X(7)**2*DF(F(19),X(6),2) - 2*X(7)*X(8)*DF(F(19),X(8),X(6))*X(3
) + 2*X(7)*X(8)*DF(F(19),X(8),2)*X(4)*X(3) - 2*X(7)*X(8)*DF(F(19)
,X(8),X(7))*X(4) + 2*X(7)*X(8)*DF(F(19),X(7),X(6)) - X(7)*DF(F(19
),X(7))*X(4) + X(7)*DF(F(19),X(8))*X(4)*X(3) + 2*X(7)*DF(F(19),X(
6),X(3))*X(4) - 2*X(7)*DF(F(19),X(8),X(3))*X(4)**2 - 2*X(7)*DF(F(
19),X(6),X(4))*X(4)*X(3) + 2*X(7)*DF(F(19),X(8),X(4))*X(4)**2*X(3
) - 2*X(7)*DF(F(19),X(8),X(1))*X(4) + 2*X(7)*DF(F(19),X(6),X(1))
 + X(8)**2*DF(F(19),X(8),2)*X(3)**2 - 2*X(8)**2*DF(F(19),X(8),X(7
))*X(3) + X(8)**2*DF(F(19),X(7),2) + 2*X(8)*DF(F(19),X(7),X(1))
 - 2*X(8)*DF(F(19),X(8))*X(4) + 2*X(8)*DF(F(19),X(7),X(3))*X(4)
 - 2*X(8)*DF(F(19),X(8),X(3))*X(4)*X(3) + 2*X(8)*DF(F(19),X(8),X(
4))*X(4)*X(3)**2 - 2*X(8)*DF(F(19),X(8),X(1))*X(3) - 2*X(8)*DF(F(
19),X(7),X(4))*X(4)*X(3) - DF(F(19),X(4))*X(4)**2 - DF(F(19),X(2)
) + DF(F(19),X(3),2)*X(4)**2 - 2*DF(F(19),X(4),X(3))*X(4)**2*X(3)
 + DF(F(19),X(4),2)*X(4)**2*X(3)**2 - 2*DF(F(19),X(4),X(1))*X(4)*
X(3)$
```

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```
LISP DEPL!*_'
(((F 34) (X 2)) ((F 28) (X 1) (X 2) (X 3) (X 4) (X 5) (X 6) (X 7)
) ((F 26) (X 2)) (REST (X 1) (X 2) (X 3) (X 4) (X 5) (X 6) (X 7)
(X 8) (X 9) (X 10) (X 11) (X 12) (X 13)));

F(3) := (2*F(28) + X(8)*DF(F(26),X(2))*X(1) + 2*X(8)*F(34) + 2*X(
9)*F(26))/2$

VER(19) := 2*DF(F(28),X(7),2)$

VER(20) :=  - 6*X(5)*F(26) + 4*X(5)*DF(F(28),X(7),X(4)) - 4*X(5)*
DF(F(28),X(7),X(6))*X(4) - 4*X(6)*DF(F(28),X(7),X(6))*X(3) + 4*X(
6)*DF(F(28),X(7),X(5)) + 4*X(7)*DF(F(28),X(7),X(6)) - 4*DF(F(28),
X(7),X(4))*X(4)*X(3) - DF(F(26),X(2))*X(1)*X(4) - DF(F(26),X(2))*
X(3) + 4*DF(F(28),X(7),X(1)) + 4*DF(F(28),X(7),X(3))*X(4) - X(1)*
DF(F(26),X(2),2) - 2*X(4)*F(34) - 2*DF(F(34),X(2))$

VER(21) := 4*DF(F(28),X(3),X(1))*X(4) + 2*DF(F(28),X(1))*X(3) + 2
*DF(F(28),X(1),2) - 4*DF(F(28),X(7))*X(5)*X(6) - 2*DF(F(28),X(7))
*X(7)*X(4) - 2*DF(F(28),X(5))*X(5)*X(4) + 4*DF(F(28),X(5),X(4))*X
(5)*X(6) - 4*DF(F(28),X(5),X(4))*X(6)*X(4)*X(3) - 2*DF(F(28),X(6)
)*X(5)**2 + 2*DF(F(28),X(6))*X(5)*X(4)*X(3) - 4*DF(F(28),X(6))*X(
6)*X(4) - 4*DF(F(28),X(6),X(4))*X(5)**2*X(4) - 4*DF(F(28),X(6),X(
4))*X(5)*X(6)*X(3) + 4*DF(F(28),X(6),X(4))*X(5)*X(7) + 4*DF(F(28)
,X(6),X(4))*X(5)*X(4)**2*X(3) + 4*DF(F(28),X(6),X(4))*X(6)*X(4)*X
(3)**2 - 4*DF(F(28),X(6),X(4))*X(7)*X(4)*X(3) + 2*X(5)**2*DF(F(28
),X(6),2)*X(4)**2 + 2*X(5)**2*DF(F(26),X(2))*X(1)*X(4) + 2*X(5)**
2*DF(F(28),X(4),2) + 4*X(5)**2*X(4)*F(34) - 4*X(5)*X(6)*DF(F(28),
X(6),X(5))*X(4) + 4*X(5)*X(6)*DF(F(28),X(6),2)*X(4)*X(3) + 2*X(5)
*X(6)*DF(F(26),X(2))*X(1)*X(3) - 4*X(5)*X(6)*DF(F(26),X(2)) + 4*X
(5)*X(6)*X(3)*F(34) - 4*X(5)*X(7)*DF(F(28),X(6),2)*X(4) - 3*X(5)*
X(7)*DF(F(26),X(2))*X(1) - 6*X(5)*X(7)*F(34) + 4*X(5)*DF(F(28),X(
4),X(3))*X(4) - 4*X(5)*DF(F(28),X(6),X(3))*X(4)**2 - 4*X(5)*DF(F(
28),X(4),2)*X(4)*X(3) + 4*X(5)*DF(F(28),X(4),X(1)) - 4*X(5)*DF(F(
28),X(6),X(1))*X(4) + 2*F(28)*X(4) - 4*X(6)**2*DF(F(28),X(6),X(5)
```

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```
)*X(3) + 2*X(6)**2*DF(F(28),X(6),2)*X(3)**2 - 2*X(6)**2*DF(F(26),
X(2))*X(1) + 2*X(6)**2*DF(F(28),X(5),2) - 4*X(6)**2*F(34) + 4*X(6
)*X(7)*DF(F(28),X(6),X(5)) - 4*X(6)*X(7)*DF(F(28),X(6),2)*X(3) -
6*X(6)*X(7)*F(26) - 4*X(6)*DF(F(28),X(6),X(3))*X(4)*X(3) - 4*X(6)
*DF(F(28),X(6),X(1))*X(3) + 4*X(6)*DF(F(28),X(5),X(3))*X(4) + 4*X
(6)*DF(F(28),X(5),X(1)) + 2*X(7)**2*DF(F(28),X(6),2) + 4*X(7)*DF(
F(28),X(6),X(3))*X(4) + X(7)*DF(F(26),X(2))*X(1)*X(4)*X(3) - 2*X(
7)*DF(F(26),X(2))*X(4) + 4*X(7)*DF(F(28),X(6),X(1)) + 2*X(7)*X(4)
*X(3)*F(34) - 2*DF(F(28),X(4))*X(4)**2 - 4*DF(F(28),X(4),X(3))*X(
4)**2*X(3) + 2*DF(F(28),X(4),2)*X(4)**2*X(3)**2 - 4*DF(F(28),X(4)
,X(1))*X(4)*X(3) + 2*X(4)**2*DF(F(28),X(3),2) - 2*DF(F(28),X(2))$
```

Appendix BURGERS 5

$$GG(10) := X*(4*U_3 + 6*U_2*U + 6*U_1^2 + 3*U_1*U^2) + 4*T*(2*U_4 + 4*U_3*U + 10*U_2*U_1 + 3*U_2*U^2 + 6*U_1^2*U + U_1*U^3) + U*(2*U_1 + U^2)$$

$$GG(11) := X^2*(U_1 + U_1*U) + X*T*(4*U_3 + 6*U_2*U + 6*U_1^2 + 3*U_1*U^2) + X*(-U_1 + U^2) + 2*T^2*(2*U_4 + 4*U_3*U + 10*U_2*U_1 + 3*U_2*U^2 + 6*U_1^2*U + U_1*U^3) + T*U*(2*U_1 + U^2) - 3*U$$

$$GG(12) := X^3*U_1 + 6*X^2*T*(U_2 + U_1*U) + 3*X^2*U + 3*X*T^2*(4*U_3 + 6*U_2*U + 6*U_1^2 + 3*U_1*U^2) + 6*X*T*(-U_1 + U^2) - 24*X + 4*T^3*(2*U_4 + 4*U_3*U + 10*U_2*U_1 + 3*U_2*U^2 + 6*U_1^2*U + U_1*U^3) + 3*T^2*U*(2*U_1 + U^2) - 18*T*U$$

$$GG(13) := X^3*T*U_1 + X^3 + 3*X^2*T^2*(U_2 + U_1*U) + 3*X^2*T*U + X*T^3*(4*U_3 + 6*U_2*U + 6*U_1^2 + 3*U_1*U^2) + 3*X*T^2*(4*U_1 + U^2) + 6*X*T + T^4*(2*U_4 + 4*U_3*U + 10*U_2*U_1 + 3*U_2*U^2 + 6*U_1^2*U + U_1*U^3) + T^3*(10*U_2 + 12*U_1*U + U^3) + 6*T^2*U$$

$$GG(14) := 2*U_4 + 4*U_3*U + 10*U_2*U_1 + 3*U_2*U^2 + 6*U_1^2*U + U_1*U^3$$

$$GG(15) := 4*X*(2*U_4 + 4*U_3*U + 10*U_2*U_1 + 3*U_2*U^2 + 6*U_1^2*U + U_1*U^3) + T*(16*U_5 + 40*U_4*U + 120*U_3*U_1 + 40*U_3*U^2 + 80*U_2^2 + 200*U_2*U_1*U + 20*U_2*U^3 + 60*U_1^3 + 60*U_1^2*U^2 + 5*U_1*U^4) + U*(4*U_2 + 6*U_1*U + U^3)$$

$$GG(16) := X^2*(4*U_3 + 6*U_2*U + 6*U_1^2 + 3*U_1*U^2) + 8*X*T*(2*U_4 + 4*U_3*U + 10*U_2*U_1 + 3*U_2*U^2 + 6*U_1^2*U + U_1*U^3) + 2*X*(-2*U_2 + U^3) + T^2*(16*U_5 + 40*U_4*U + 120*U_3*U_1 + 40*U_3*U^2 + 80*U_2^2 + 200*U_2*U_1*U + 20*U_2*U^3 + 60*U_1^3 + 60*U_1^2*U^2 + 5*U_1*U^4) + 2*T*U*(4*U_2 + 6*U_1*U + U^3) - 6*U^2$$

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$$GG(17) := 2*X^3*(U_2 + U_1*U) + 3*X^2*T*(4*U_3 + 6*U_2*U + 6*U_1^2 + 3*U_1*U^2) + 3*X^2*(- 2*U_1 + U^2) + 12*X*T^2*(2*U_4 + 4*U_3*U + 10*U_2*U_1 + 3*U_2*U^2 + 6*U_1^2*U + U_1*U^3) + 6*X*T*(- 2*U_2 + U^3) - 24*X*U + T^3*(16*U_5 + 40*U_4*U + 120*U_3*U_1 + 40*U_3*U^2 + 80*U_2^2 + 200*U_2*U_1*U + 20*U_2*U^3 + 60*U_1^3 + 60*U_1^2*U^2 + 5*U_1*U^4) + 3*T^2*U*(4*U_2 + 6*U_1*U + U^3) - 18*T*U^2 + 60$$

$$GG(18) := X^4*U_1 + 8*X^3*T*(U_2 + U_1*U) + 4*X^3*U + 6*X^2*T^2*(4*U_3 + 6*U_2*U + 6*U_1^2 + 3*U_1*U^2) + 12*X^2*T*(- 2*U_1 + U^2) - 60*X^2 + 16*X*T^3*(2*U_4 + 4*U_3*U + 10*U_2*U_1 + 3*U_2*U + 6*U_1^2*U + U_1*U^3) + 12*X*T^2*(- 2*U_2 + U^3) - 96*X*T*U + T^4*(16*U_5 + 40*U_4*U + 120*U_3*U_1 + 40*U_3*U^2 + 80*U^2 + 200*U_2*U_1*U + 20*U_2*U^3 + 60*U_1^3 + 60*U_1^2*U^2 + 5*U_1*U^4) + 4*T^3*U*(4*U^2 + 6*U_1*U + U^3) + 36*T^2*(- 5*U_1 - U^2) - 120*T$$

$$GG(19) := 5*X^4*T*U_1 + 5*X^4 + 20*X^3*T^2*(U_2 + U_1*U) + 20*X^3*T*U + 10*X^2*T^3*(4*U_3 + 6*U_2*U + 6*U_1^2 + 3*U_1*U^2) + 30*X^2*T^2*(4*U_1 + U^2) + 60*X^2*T + 20*X*T^4*(2*U_4 + 4*U_3*U + 10*U_2*U_1 + 3*U_2*U^2 + 6*U_1^2*U + U_1*U^3) + 20*X*T^3*(10*U_2 + 12*U_1*U + U^3) + 120*X*T^2*U + T^5*(16*U_5 + 40*U_4*U + 120*U_3*U_1 + 40*U_3*U^2 + 80*U_2^2 + 200*U_2*U_1*U + 20*U_2*U^3 + 60*U_1^3 + 60*U_1^2*U^2 + 5*U_1*U^4) + 5*T^4*(24*U_3 + 40*U_2*U + 36*U_1^2 + 24*U_1*U^2 + U^4) + 60*T^3*(3*U_1 + U^2) + 60*T^2$$

$$G(20) := 16*U_5 + 40*U_4*U + 120*U_3*U_1 + 40*U_3*U^2 + 80*U_2^2 + 200*U_2*U_1*U + 20*U_2*U^3 + 60*U_1^3 + 60*U_1^2*U^2 + 5*U_1*U^4$$

4.2 Lie-Bäcklund transformations of the Classical Boussinesq equation

The Classical Boussinesq Equation is written as the following system of partial differential equations

$$u_t = (uv + \sigma v_{xx})_x \qquad (\sigma \text{ parameter})$$

$$v_t = (u + \tfrac{1}{2} v^2)_x. \tag{4.2.1}$$

In a recent paper Ito [14] showed that for this equation there exists a matrix recursion operator $\mathcal{D}$, defined by

$$\mathcal{D} = \begin{bmatrix} \frac{1}{2} v & \sigma D^2 + u + \frac{1}{2} u_x D^{-1} \\ 1 & \frac{1}{2} v + \frac{1}{2} v_x D^{-1} \end{bmatrix}, \tag{4.2.2}$$

which generates an infinite hierarchy of commuting Lie-Bäcklund transformations.
In (4.2.2) D is defined by (1.2.5)

$$D = D_1 = \partial_x + u_x \partial_u + v_x \partial_v + \ldots \tag{4.2.3a}$$

and D^{-1} by

$$D^{-1} f = \int_{-\infty}^{x} f \, dx, \text{ where } f = f(x,t,u,u_x,u_t,\ldots) \tag{4.2.3b}$$

(c.f. [14], [28]).

This means that, given a Lie-Bäcklund transformation

$$X_j = \begin{bmatrix} H_j \\ K_j \end{bmatrix} = H_j \partial_u + K_j \partial_v, \tag{4.2.4}$$

then

$$X_{j+1} = \begin{bmatrix} H_{j+1} \\ X_{j+1} \end{bmatrix} = \mathcal{D} X_j \tag{4.2.5}$$

is a Lie-Bäcklund transformation.

In (4.3.4) only the ∂_u, ∂_v components are given; the other components are obtained by total partial differentation (1.2.14).
We shall derive a finite number of Lie-Bäcklund transformations of (4.2.1) by symbolic integration.

Due to this derivation we obtain an (x,t)-dependent Lie-Bäcklund transformation, which does have a similar action on the (x,t)-independent ones as the matrix recursion operator $\mathcal{D}$ (4.2.2), and which is probably a generating Lie-Bäcklund transformation.

First of all we note that equations (4.2.1) can be graded in the following way [39]

$$\begin{aligned} &\deg(u) = 1,\ \deg(x) = -\deg(\partial_x) = -\tfrac{1}{2} \\ &\deg(v) = \tfrac{1}{2},\ \deg(t) = -\deg(\partial_t) = -1 \end{aligned} \qquad (4.2.6)$$

In order to search for Lie-Bäcklund transformation of (4.2.1) we introduce the ideal I of differential 1-forms defined on

$\mathbb{R}^{13} = \{(x(1),\dots,x(13))\} = \{(x,t,u,v,u_{01},v_1,v_{01},v_2,v_{11},v_{02},v_{21},v_{12},v_{03})\}$
generated by

$$\begin{aligned} \alpha_1 &= du - u_1dx - u_{01}dt \\ \alpha_2 &= dv - v_1dx - v_{01}dt \\ \alpha_3 &= dv_1 - v_2dx - v_{11}dt \\ \alpha_4 &= dv_{01} - v_{11}dx - v_{02}dt \\ \alpha_5 &= dv_2 - v_3dx - v_{21}dt \\ \alpha_6 &= dv_{11} - v_{21}dx - v_{12}dt \\ \alpha_7 &= dv_{02} - v_{12}dx - v_{03}dt \end{aligned} \qquad (4.2.7)$$

while u_1, v_3 are obtained from (4.2.1). In (4.2.7) a notation is used similar to (4.1.8). We furthermore construct the 3 times prolonged ideal D^3I defined on

$$\mathbb{R}^{25} = \{(x,t,u,v,u_{01},v_1,v_{01},v_2,v_{11},v_{02},v_{21},v_{12},v_{03}, u_{02},v_{22},v_{13},v_{04},u_{03},v_{23},v_{14},v_{05},u_{04},v_{24},v_{15},v_{06})\}$$

and generated by

$$
\begin{aligned}
&\alpha_1, \ldots, \alpha_7 \quad (4.2.7)\\
&\alpha_8 = du_{01} - u_{11}\,dx - u_{02}\,dt\\
&\alpha_9 = dv_{21} - u_{31}\,dx - v_{22}\,dt\\
&\alpha_{10} = dv_{12} - v_{22}\,dx - v_{13}\,dt\\
&\alpha_{11} = dv_{03} - v_{13}\,dx - v_{04}\,dt\\
&\alpha_{12} = du_{02} - u_{12}\,dx - u_{03}\,dt\\
&\alpha_{13} = dv_{22} - v_{32}\,dx - v_{23}\,dt\\
&\alpha_{14} = dv_{13} - v_{23}\,dx - v_{14}\,dt\\
&\alpha_{15} = dv_{04} - v_{14}\,dx - v_{05}\,dt\\
&\alpha_{16} = du_{03} - u_{13}\,dx - u_{04}\,dt\\
&\alpha_{17} = dv_{23} - v_{33}\,dx - v_{24}\,dt\\
&\alpha_{18} = dv_{14} - v_{24}\,dx - v_{15}\,dt\\
&\alpha_{19} = dv_{05} - v_{15}\,dx - v_{06}\,dt,
\end{aligned}
\tag{4.2.8}
$$

where u_{11}, v_{31}, u_{12}, v_{32}, u_{13}, v_{33} can be obtained from (4.2.1) by total differentiation with respect to t,

$$
\begin{aligned}
u_1 &= v_{01} - v\,v_1\\
u_{11} &= v_{02} - v_{01}v_1 - v\,v_{11}\\
u_{12} &= v_{03} - v_{02}v_1 - 2v_{01}v_{11} - v\,v_{12}\\
u_{13} &= v_{04} - v_{03}v_1 - 3v_{02}v_{11} - 3v_{01}v_{12} - v\,v_{13}\\
\sigma v_3 &= u_{01} - u_1 v - u\,v_1\\
\sigma v_{31} &= u_{02} - u_{11}v - u_1 v_{01} - u_1 v_1 - u\,v_{11}\\
\sigma v_{32} &= u_{03} - u_{12}v - 2u_{11}v_{01} - u_1 v_{02}\\
&\quad - u_{02}v_1 - 2u_{01}v_{11} - u\,v_{12}\\
\sigma v_{33} &= u_{04} - u_{13}v - 3u_{12}v_{01} - 3u_{11}v_{02} - u_1 v_{03}\\
&\quad - u_{03}v_1 - 3u_{02}v_{11} - 3u_{01}v_{12} - u\,v_{13}
\end{aligned}
\tag{4.2.9}
$$

Now let V be the vertical vector field defined on $\mathbb{R}^{25}$ by

$$V = F(3)\partial_{x(3)} + F(4)\partial_{x(4)} + \ldots \tag{4.2.10}$$

Note that in (4.2.10)

$$\partial_{x(3)} = \partial_u \;;\; \partial_{x(4)} = \partial_v. \tag{4.2.10a}$$

We suppose F(3), F(4) to be dependent on x,t,u,v,...; up to and including third order derivatives in v, and up to and including second order derivatives in u, i.e.,

$$x,t,u,v,u_{01},v_1,v_{01},v_2,v_{11},v_{02},v_{21},v_{12},v_{03},u_{02}. \qquad (4.2.11)$$

The condition (1.2.17) on V,

$$L_V\ (I) \subset D^3 I \qquad (4.2.12)$$

results in an overdetermined system of 14 partial differential equations.

We solved this overdetermined system by symbolic integration. In the solution procedure we took advantage of the grading of the Boussinesq Equation (4.2.1), (4.2.6).

The complete solution of (4.2.10-12) consists of linear combinations of 8 vector fields $Y_1,...,Y_8$.

Transformation of the variables (4.2.11) into x-derivatives of u, v due to the fact that (4.2.1) is an evolution equation,

$$\begin{aligned}
&x = x,\ t = t,\ u = u,\ v = v,\ v_1 = v_1,\ v_2 = v_2\\
&u_{01} = \sigma v_3 + u_1 v + v_1 u\\
&v_{01} = u_1 + v_1 v\\
&v_{11} = u_2 + v_2 v + v_1^2\\
&v_{02} = \sigma v_4 + 2u_2 v + v_2(v^2+u) + 3u_1 v_1 + 2v_1^2 v \qquad (4.2.13)\\
&v_{21} = u_3 + v_3 v + 3v_2 v_1\\
&v_{12} = \sigma v_5 + 2u_3 v + v_3(v^2+u) + 5u_2 v_1 + 4v_2 u_1 + 6v_2 v_1 v + 2v_1^3\\
&v_{03} = \sigma u_5 + 3\sigma v_5 v + 8\sigma v_4 v_1 + u_3(3v^2+u) + 11\sigma v_3 v_2\\
&\qquad + v_3 v(v^2+3u) + 5u_2 u_1 + 15u_2 v_1 v + 12v_2 u_1 v +\\
&\qquad + v_2 v_1(9v^2+7u) + 11u_1 v_1^2 + 6v_1^3 v\\
&u_{02} = \sigma u_4 + 2\sigma v_4 v + 5\sigma v_3 v_1 + u_2(v^2+u) + 3\sigma v_2 +\\
&\qquad + 2v_2 vu + u_1^2 + 4u_1 v_1 v + 2v_1^2 u
\end{aligned}$$

leads to the result,

$$Y_i = Y(i,3)\partial_u + Y(i,4)\partial_v + \dots \qquad (i=1,..,8)$$

$$Y(1,3) := V_3 \ast \mathrm{SIG} + U_1 \ast V + V_1 \ast U \qquad (\mathrm{SIG}=\sigma)$$

$$Y(1,4) := U_1 + V_1 \ast V$$

$$Y(2,3) := U_1$$

$$Y(2,4) := V_1$$

$$Y(3,3) := T \ast U_1$$

$$Y(3,4) := T \ast V_1 + 1$$

$$Y(4,3) := (X \ast U_1 + 2 \ast T \ast V_3 \ast \mathrm{SIG} + 2 \ast T \ast U_1 \ast V + 2 \ast T \ast V_1 \ast U + 2 \ast U)/2$$

$$Y(4,4) := (X \ast V_1 + 2 \ast T \ast U_1 + 2 \ast T \ast V_1 \ast V + V)/2$$

$$Y(5,3) := (2 \ast X \ast V_3 \ast \mathrm{SIG} + 2 \ast X \ast U_1 \ast V + 2 \ast X \ast V_1 \ast U + 4 \ast T \ast U_3 \ast \mathrm{SIG} + 6 \ast T \ast V_3 \ast \mathrm{SIG} \ast V + 12 \ast T \ast V_2 \ast V_1 \ast \mathrm{SIG} + 3 \ast T \ast U_1 \ast (V^2 + 2 \ast U) + 6 \ast T \ast V_1 \ast V \ast U + 6 \ast V^2 \ast \mathrm{SIG} + 4 \ast V \ast U)/(4 \ast \mathrm{SIG})$$

$$Y(5,4) := (2 \ast X \ast U_1 + 2 \ast X \ast V_1 \ast V + 4 \ast T \ast V_3 \ast \mathrm{SIG} + 6 \ast T \ast U_1 \ast V + 3 \ast T \ast V_1 \ast (V^2 + 2 \ast U) + V^2 + 4 \ast U)/(4 \ast \mathrm{SIG})$$

$$Y(6,3) := (2 \ast V_5 \ast \mathrm{SIG}^2 + 4 \ast U_3 \ast \mathrm{SIG} \ast V + V_3 \ast \mathrm{SIG} \ast (3 \ast V^2 + 5 \ast U) + 9 \ast U_2 \ast V_1 \ast \mathrm{SIG} + 10 \ast V_2 \ast U_1 \ast \mathrm{SIG} + 12 \ast V_2 \ast V_1 \ast \mathrm{SIG} \ast V + U_1 \ast V \ast (V^2 + 6 \ast U) + 3 \ast V_1^3 \ast \mathrm{SIG} + 3 \ast V_1 \ast U \ast (V^2 + U))/\mathrm{SIG} \qquad (4.2.14)$$

$$Y(6,4) := (2 \ast U_3 \ast \mathrm{SIG} + 4 \ast V_3 \ast \mathrm{SIG} \ast V + 7 \ast V_2 \ast V_1 \ast \mathrm{SIG} + 3 \ast U_1 \ast (V^2 + U) + V_1 \ast V \ast (V^2 + 6 \ast U))/\mathrm{SIG}$$

$$Y(7,3) := (16 \ast U_5 \ast \mathrm{SIG}^2 + 40 \ast V_5 \ast \mathrm{SIG}_2 \ast V + 120 \ast V_4 \ast V_1 \ast \mathrm{SIG}^2 + 40 \ast U_3 \ast \mathrm{SIG} \ast (V^2 + U) + 200 \ast V_3 \ast V_2 \ast \mathrm{SIG}^2 + 20 \ast V_3 \ast \mathrm{SIG} \ast V \ast (V^2 + 5 \ast U) + 80 \ast U_2 \ast U_1 \ast \mathrm{SIG} + 180 \ast U_2 \ast V_1 \ast \mathrm{SIG} \ast V + 200 \ast V_2 \ast U_1 \ast \mathrm{SIG} \ast V + 40 \ast V_2 \ast V_1 \ast \mathrm{SIG} \ast (3 \ast V^2 + 5 \ast U) + 150 \ast U_1 \ast V_1^2 \ast \mathrm{SIG} + 5 \ast U_1 \ast (V^4 + 12 \ast V^2 \ast U + 6 \ast U^2) + 60 \ast V_1^3 \ast \mathrm{SIG} \ast V + 20 \ast V_1 \ast V \ast U \ast (V^2 + 3 \ast U))/(16 \ast \mathrm{SIG})$$

$$Y(7,4) := (16*V_5*SIG^2 + 40*U_3*SIG*V + 40*V_3*SIG*(V^2 + U) + 80*U_2*V_1*SIG + 80*V_2*U_1*SIG + 140*V_2*V_1*SIG*V + 20*U_1*V*(V^2 + 3*U) + 30*V_1^3*SIG + 5*V_1*(V^4 + 12*V^2*U + 6*U^2))/(16*SIG)$$

$$Y(8,3) := (4*U_3*SIG + 6*V_3*SIG*V + 12*V_2*V_1*SIG + 3*U_1*(V^2 + 2*U) + 6*V_1*V*U)/(4*SIG)$$

$$Y(8,4) := (4*V_3*SIG + 6*U_1*V + 3*V_1*(V^2 + 2*U))/(4*SIG)$$

The Lie bracket of the vertical vector fields Y_i $(i=1,\dots,8)$ is defined by

$$[Y_i,Y_j]^{\ell} = L_{Y_i}(Y(j,\ell)) - L_{Y_j}(Y(i,\ell)) \quad (\ell=3,4) \tag{4.2.15}$$

The commutators of the vector fields are given in the following table

j →

$[Y_i,Y_j]$ (i↓)	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8
Y_1	0	0	$-Y_2$	$-Y_1$	$-Y_8$	0	0	0
Y_2		0	0	$-\frac{1}{2}Y_2$	$-\frac{1}{2}\sigma^{-1}Y_4$	0	0	0
Y_3			0	$\frac{1}{2}Y_3$	$\sigma^{-1}Y_4$	$4Y_8$	$\frac{5}{4}Y_6$	$\frac{3}{2}\sigma^{-1}Y_1$
Y_4				0	$\frac{1}{2}Y_5$	$2Y_6$	$\frac{5}{2}Y_7$	$\frac{3}{2}Y_8$
Y_5					0	$4\sigma^{-1}Y_7$	Y_9	$\frac{3}{4}\sigma^{-1}Y_6$
Y_6						0	0	0
Y_7							0	0
Y_8								0

table 4.2.1

In table 4.2.1 Y_9 is defined by

$$\begin{aligned} Y(9,3) := {} & (80*V_7*SIG^3 + 240*U_5*SIG^2*V + 20*V_5*SIG^2*(15*V^2 + 14*U \\ &) + 800*U_4*V_1*SIG^2 + 840*V_4*U_1*SIG^2 + 1800*V_4*V_1* \\ & SIG^2*V + 1400*U_3*V_2*SIG^2 + 200*U_3*SIG*V*(V^2 + 3*U) + \\ & 1400*V_3*U_2*SIG^2 + 3000*V_3*V_2*SIG^2*V + 2250*V_3*V_1^2* \\ & SIG^2 + 25*V_3*SIG*(3*V^4 + 30*V^2*U + 14*U^2) + 1200*U_2* \\ & U_1*SIG*V + 50*U_2*V_1*SIG*(27*V^2 + 26*U) + 3000*V_2^2*V_1* \\ & SIG^2 + 100*V_2*U_1*SIG*(15*^2 + 14*U) + 600*V_2*V_1*SIG*V \\ & *(V^2 + 5*U)+ 1000*U_1^2*V_1*SIG + 2250*U_1*V_1^2*SIG*V + 15* \\ & U_1*V*(V^4 + 20*V^2*U + 30*U^2) + 150*V_1^3*SIG*(3*V^2 + 5*U) \\ & + 75*V_1*U*(V^4 + 6*V^2*U_2 + 2*U^2))/(32*SIG^2) \end{aligned} \qquad (4.2.16)$$

$$\begin{aligned} Y(9,4) := {} & (80*U_5*SIG^2 + 240*V_5*SIG^2*V + 640*V_4*V_1*SIG^2 + 100*U_3* \\ & SIG*(3*V^2 + 2*U) + 1000*V_3*V_2*SIG^2 + 200*V_3*SIG*V*(\\ & V^2 + 3*U) + 400*U_2*U_1*SIG + 1200*U_2*V_1*SIG*V + \\ & 1200*V_2*U_1*SIG*V + 50*V_2*V_1*SIG*(21*V^2 + 22*U) + 850* \\ & U_1*V_1^2*SIG + 75*U_1*(V^4 + 6*V^2*U + 2*U^2) + 450*V_1^3*SIG*V \\ & + 15*V_1*V*(V^4 + 20*V^2*U + 30*U^2))/(32*SIG^2) \end{aligned}$$

In order to transform the Lie algebra we introduce

$$\begin{array}{lll} Z_1 = \sigma Y_5 & W_1 = Y_2 & W_4 = \frac{3}{8}\sigma Y_6 \\ Z_0 = Y_4 & W_2 = \frac{1}{2} Y_1 & W_5 = \frac{3}{2}\sigma Y_7 \\ Z_{-1} = Y_3 & W_3 = \frac{\sigma}{2} Y_8 & W_6 = \frac{3}{2}\sigma^2 Y_9, \end{array} \qquad (4.2.17)$$

which results in the following Lie commutator table

[*,*]	Z_1	Z_0	Z_{-1}	W_1	W_2	W_3	W_4	W_5
Z_1	0	$-\frac{1}{2}Z_1$	$-Z_0$	W_2	W_3	W_4	W_5	W_6
Z_0		0	$\frac{1}{2}Z_{-1}$	$\frac{1}{2}W_1$	W_2	$\frac{3}{2}W_3$	$2W_4$	$\frac{5}{2}W_5$
Z_{-1}			0	0	$\frac{1}{2}W_1$	$\frac{3}{2}W_2$	$3W_3$	$5W_4$
W_1				0	0	0	0	0
W_2					0	0	0	0
W_3						0	0	0
W_4							0	0
W_5								0

table 4.2.2

From the results in table 4.2.2 we formulate the following

<u>Conjecture</u>

The vector fields Z_1, Z_0, Z_{-1}, W_1, generate an infinite-dimensional Lie algebra of Lie-Bäcklund transformations of the Classical Bousinesq equation (4.2.1) of the following structure

$$
\begin{aligned}
[Z_1,W_i] &= W_{i+1} \quad (i=1,\ldots.), \; [W_i,W_j] = 0 \; (i,j=1,\ldots) \\
[Z_0,W_i] &= \frac{i}{2} W_i \; (i=1,\ldots.) \\
[Z_{-1},W_i] &= \frac{1}{4} i(i-1)W_{i-1} \; (i=1,\ldots.) \\
[Z_1,Z_0] &= -\frac{1}{2} Z_1, \; [Z_1,Z_{-1}] = -Z_0, \; [Z_0,Z_{-1}] = -\frac{1}{2} Z_{-1},
\end{aligned}
\tag{4.2.18}
$$

where W_i (i=1,...) are (x,t)-independent.

4.3 Lie-Bäcklund transformations of the Massive Thirring Model

We shall establish the existence of Lie-Bäcklund transformations of the Massive Thirring Model, which can be defined as the following system of partial differential equations for the unknown functions $u_1(x,t), v_1(x,t), u_2(x,t), v_2(x,t)$ [15]

$$\begin{aligned}
-\frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial t} &= mv_2 - (u_2^2+v_2^2)v_1 \\
\frac{\partial u_2}{\partial x} + \frac{\partial u_2}{\partial t} &= mv_1 - (u_1^2+v_1^2)v_2 \\
\frac{\partial v_1}{\partial x} - \frac{\partial v_1}{\partial t} &= mu_2 - (u_2^2+v_2^2)u_1 \\
-\frac{\partial v_2}{\partial x} - \frac{\partial v_2}{\partial t} &= mu_1 - (u_1^2+v_1^2)u_2
\end{aligned} \tag{4.3.1}$$

The ideal I in $\mathbb{R}^{10} = \{(x,t,u_1,u_2,v_1,v_2,u_{1x},u_{2x},v_{1x},v_{2x})\}$ describing (4.3.1) is generated by the 4 1-forms

$$\begin{aligned}
\alpha_1 &= du_1 - u_{1x}dx - G(11)dt \\
\alpha_2 &= du_2 - u_{2x}dx - G(21)dt \\
\alpha_3 &= dv_1 - v_{1x}dx - G(31)dt \\
\alpha_4 &= dv_2 - v_{2x}dx - G(41)dt
\end{aligned} \tag{4.3.2}$$

where $G(11),\dots,G(41)$ are defined by

$$\begin{aligned}
G(11) &= u_{1x} + mv_2 - (u_2^2+v_2^2)v_1 \\
G(21) &= -u_{2x} + mv_1 - (u_1^2+v_1^2)v_2 \\
G(31) &= v_{1x} - mu_2 + (u_2^2+v_2^2)u_1 \\
G(41) &= -v_{2x} - mu_1 + (u_1^2+v_1^2)u_2
\end{aligned} \tag{4.3.3}$$

The 2 times prolonged ideal D^2I in $\mathbb{R}^{18}$ is given by

$$\begin{aligned}
&\alpha_1,\alpha_2,\alpha_3,\alpha_4 \ (4.3.2) \\
\alpha_5 &= du_{1x} - u_{1xx}dx - G(12)dt \\
\alpha_6 &= du_{2x} - u_{2xx}dx - G(22)dt \\
\alpha_7 &= dv_{1x} - v_{1xx}dx - G(32)dt
\end{aligned}$$

$$\begin{aligned}
\alpha_8 &= dv_{2x} - v_{2xx}dx - G(42)dt \\
\alpha_9 &= du_{1xx} - u_{1xxx}dx - G(13)dt \\
\alpha_{10} &= du_{2xx} - u_{2xxx}dx - G(23)dt \\
\alpha_{11} &= dv_{1xx} - v_{1xxx}dx - G(33)dt \\
\alpha_{12} &= dv_{2xx} - v_{2xxx}dx - G(43)dt
\end{aligned} \tag{4.3.4}$$

where the coefficients G(**) are derived by total partial differentiation of G(11),G(21),G(31),G(41) with respect to x.
Now the vector field

$$V = F(3)\partial_{u_1} + F(4)\partial_{u_2} + F(5)\partial_{v_1} + F(6)\partial_{v_2} \tag{4.3.5}$$

is a Lie-Bäcklund transformation for (4.3.1) if

$$\mathcal{L}_V I \subset D^2 I, \tag{4.3.6}$$

which leads to an overdetermined system of 2 x 4 = 8 partial differential equations for F(3),F(4),F(5),F(6) the defining coefficients of the vector field V (4.3.5).

If we want to search for the vertical vector fields equivalent to the classical symmetries of (4.2.1) we require F(3),...,F(6) to be dependent on $x,t,u_1,\ldots,v_{2x}$.
A straightforward computation then leads to the following 4 symmetries

$$X_i = X_i^3\partial_{u_1} + X_i^4\partial_{u_2} + X_i^5\partial_{v_1} + X_i^6\partial_{v_2} \qquad (i=1,\ldots,4)$$

where

$$\begin{aligned}
X_1^3 &= (\tfrac{1}{2})(-mv_2+v_1(u_2^2+v_2^2)) \\
X_1^4 &= (\tfrac{1}{2})(2u_{2x}-mv_1+v_2(u_1^2+v_1^2)) \\
X_1^5 &= (\tfrac{1}{2})(mu_2-u_1(u_2^2+v_2^2)) \\
X_1^6 &= (\tfrac{1}{2})(2v_{2x}+mu_1-u_2(u_1^2+v_1^2))
\end{aligned} \tag{4.3.7}$$

$$
\begin{aligned}
X_2^3 &= (\tfrac{1}{2})(2u_{1x}+mv_2-v_1(u_2^2+v_2^2)) \\
X_2^4 &= (\tfrac{1}{2})(mv_1-v_2(u_1^2+v_1^2)) \\
X_2^5 &= (\tfrac{1}{2})(2v_{1x}-mu_2+u_1(u_2^2+v_2^2)) \\
X_2^6 &= (\tfrac{1}{2})(-mu_1+u_2(u_1^2+v_1^2)) \\
X_3^3 &= u_{1x}(x+t) + mv_2x + (\tfrac{1}{2})u_1 - v_1(u_2^2+v_2^2)x \\
X_3^4 &= u_{2x}(-x+t) + mv_1x - (\tfrac{1}{2})u_2 - v_2(u_1^2+v_1^2)x \\
X_3^5 &= v_{1x}(x+t) - mu_2x + (\tfrac{1}{2})v_1 + u_1(u_2^2+v_2^2)x \\
X_3^6 &= v_{2x}(-x+t) - mu_1x - (\tfrac{1}{2})v_2 + u_2(u_1^2+v_1^2)x \\
X_4^3 &= v_1;\ X_4^4 = v_2;\ X_4^5 = -u_1;\ X_4^6 = -u_2.
\end{aligned}
\tag{4.3.7}
$$

In order to find Lie-Bäcklund transformations for (4.3.1) we introduce a grading in the following way

$$
\begin{aligned}
\deg(x) &= -2 \\
\deg(t) &= -2 \\
\deg(u_1) &= \deg(u_2) = \deg(v_1) = \deg(v_2) = 1 \\
\deg(m) &= 2.
\end{aligned}
\tag{4.3.8}
$$

By (4.3.8) each term in (4.3.1) is of degree 3.
Now, since the total partial differentiation operators are graded $\deg(D_x) = \deg(D_t) = 2$, solutions of the Lie-Bäcklund symmetry condition are graded correspondingly.

We introduce the following notation:

$$
\begin{aligned}
&[u] \text{ refers to } u_1, u_2, v_1 \text{ or } v_2 \\
&[u]_x \text{ refers to } u_{1x}, u_{2x}, v_{2x} \text{ or } v_{2x}.
\end{aligned}
\tag{4.3.9}
$$

In the search for Lie-Bäcklund transformations we did not construct the general solution of the overdetermined system of partial differential equations. We restrict our search to those induced by (4.3.8), which motivated us to seek a solution of the following form

$$F(J) := [u]_{xx} + ([u]^2+[m])[u]_x + ([u]^5+[m][u]^3+[m]^2[u]) \quad (J=3,\ldots,6) \qquad (4.3.10)$$

i.e. a vector field whose defining coefficients are of degree 5.
In fact, we only introduced the maximal power of the coefficient of $[u]_x$ i.e. $[u]^2$, and the maximal power of $[u]$, i.e. $[u]^5$ into the overdetermined system of partial differential equations.

Using the integration package, we found 2 Lie-Bäcklund transformations X_5, X_6 given by

$$\begin{aligned}
X_5^3 &= \tfrac{1}{4}\{2u_{2x}(-m+2v_1v_2) - 4v_{2x}u_2v_1 - mv_2(R_1+R_2) \\
&\quad - 2mv_1R + v_1(R_2^2+2R_1R_2)\} \\
X_5^4 &= \tfrac{1}{4}\{-4v_{2xx} + 2u_{1x}(-m+2u_1u_2) + 4u_{2x}(R_1+R_2) + 4v_{1x}u_2v_1 \\
&\quad - mv_1(R_1 + R_2) - 2mv_2R + v_2(R_1^2+2R_1R_2)\} \\
X_5^5 &= \tfrac{1}{4}\{2v_{2x}(-m+2u_1u_2) - 4u_{2x}u_1v_2 + mu_2(R_1+R_2) \\
&\quad + 2mu_1R - u_1(R_2^2+2R_1R_2)\} \\
X_5^6 &= \tfrac{1}{4}\{4u_{2xx}+ 2v_{1x}(-m+2v_1v_2) + 4v_{2x}(R_1+R_2) + 4u_{1x}u_1v_2 \\
&\quad + mu_1(R_1+R_2) + 2mu_2R - u_2(R_1^2+2R_1R_2)\}
\end{aligned} \qquad (4.3.11a)$$

and

$$\begin{aligned}
X_6^3 &= \tfrac{1}{4}\{4v_{1xx} + 2u_{2x}(-m+2u_1u_2) + 4u_{1x}(R_1+R_2) + 4v_{2x}u_1v_2 \\
&\quad + mv_2(R_1+R_2) + 2mv_1R - v_1(R_2^2+2R_1R_2)\} \\
X_6^4 &= \tfrac{1}{4}\{2u_{1x}(-m+2\ {}_1v_2) - 4v_{1x}u_1v_2 + mv_1(R_1+R_2) \\
&\quad + 2mv_2R - v_2(R_1^2+2R_1R_2)\} \\
X_6^5 &= \tfrac{1}{4}\{-4u_{1xx} + 2v_{2x}(-m+2v_1v_2) + 4v_{1x}(R_1+R_2) + 4u_{2x}u_2v_1 \\
&\quad - mu_2(R_1+R_2) - 2mu_1R + u_1(R_2^2+2R_1R_2)\} \\
X_6^6 &= \tfrac{1}{4}\{2v_{1x}(-m+2u_1u_2) - 4u_{1x}u_2v_1 - mu_1(R_1+R_2) \\
&\quad - 2mu_2R + u_2(R_1^2+2R_1R_2)\}
\end{aligned} \qquad (4.3.11b)$$

whereas in (4.3.11a,b)

$$R = u_1u_2 + v_1v_2,\ R_1 = u_1^2 + v_1^2,\ R_2 = u_2^2 + v_2^2. \tag{4.3.12}$$

In order to find third order Lie-Bäcklund transformation we have to prolong the ideal D^2I once more.

In the search for third order results we restricted ourselves to vector fields whose defining coefficients are schematically given by

$$\begin{aligned} F(J) = {} & [u]_{xxx} + ([u]^2+[m])[u]_{xx} + [u][u]_x^2 \\ & ([u]^4+[u]^2[m] + [m]^2)[u]_x + \\ & ([u]^7+[u]^5[m]+[u]^3[m]^2+[u][m]^3). \quad (J:=3:6) \end{aligned} \tag{4.3.13}$$

After a massive amount of computations we obtained two additional Lie Bäcklund transformations X_7, X_8 i.e.

$$\begin{aligned} X_7^3 = {} & (1/8)\{8u_{2xx}u_2v_1 + 4v_{2xx}(2v_1v_2-m) - 4\ u_{2x}^2v_1 + 4u_{2x}(m(R_1+R_2+v_1^2+v_2^2) \\ & - 3v_1v_2(R_1+R_2)) - 4v_{2x}^2v_1 + 4v_{2x}(-(u_1v_1+u_2v_2)m + 3u_2v_1(R_1+R_2)) \\ & + 4u_{1x}mR - 2m^2v_1(R_1+R_2) - 4v_2m^2R + 4v_1mR(R_1+2R_2) \\ & + v_2m(R_1^2+4R_1R_2+R_2^2) - v_1(R_2^3+6R_2^2R_1+3R_2R_1^2)\} \\ X_7^4 = {} & (1/8)\{8u_{2xxx} + 12v_{2xx}(R_1+R_2) + 8u_{1xx}u_1v_2 + 4v_{1xx}(2v_1v_2-m) - 12u_{2x}^2v_2 \\ & + 24u_{2x}v_{2x}u_2 + 2u_{2x}(10mR-3R_1^2-12R_1R_2-3R_2^2) + 12v_{2x}^2v_2 + 24v_{2x}u_{1x}u_1 \\ & + 24v_{2x}v_{1x}v_1 + 8u_{1x}^2v_2 + 4u_{1x}(m(R_1+R_2+u_1^2+u_2^2)-3u_1u_2(R_1+R_2)) \\ & + 8v_{1x}^2v_2 + 4v_{1x}(m(u_1v_1+u_2v_2) - 3u_2v_1(R_1+R_2)) - 4m^2v_1R - 2m^2v_2(R_1+R_2) \\ & + mv_1(R_2^2+4R_1R_2-R_1^2) + 4mv_2R(R_2+2R_1) - v_2(R_1^3+6R_1^2R_2+3R_1R_2^2)\} \end{aligned} \tag{4.3.14}$$

$$\begin{aligned} X_7^5 = {} & (1/8)\{-8v_{2xx}v_2u_1 - 4v_{2xx}(u_1u_2-m) + 4v_{2x}^2u_1+4v_{2x}(m(R_1+R_2+u_1^2+u_2^2)+ \\ & - 3u_1u_2(R_1+R_2)) + 4u_{2x}^2u_1 + 4u_{2x}(-(u_1v_1+u_2v_2)m + 3v_2u_1(R_1+R_2)) \\ & +4v_{1x}mR + 2m^2u_1(R_1+R_2) + 4u_2m^2R - 4u_1mR(R_1+2R_2) \\ & - u_2m(R_1^2+4R_1R_2+R_2^2) + u_1(R_2^3+6R_2^2R_1+3R_2R_1^2)\} \end{aligned}$$

$$
\begin{aligned}
X_7^6 = (1/8)\{&8v_{2xxx} - 12u_{2xx}(R_1+R_2) + 8v_{1xx}u_2v_1 - 4u_{1xx}(2u_1u_2-m) + 12v_{2x}^2u_2 \\
&- 24u_{2x}v_{2x}v_2 + 2v_{2x}^2(10mR-3R_1^2-12R_1R_2-3R_2^2) - 12u_{2x}^2u_2 - 24u_{2x}v_{1x}v_1 \\
&+ 24u_{2x}u_{1x}u_1 - 8v_{1x}^2u_2 + 4v_{1x}(m(R_1+R_2+v_1^2+v_2^2) - 3v_1v_2(R_1+R_2)) \\
&- 8u_{1x}^2u_2 + 4u_{1x}(m(u_1v_1+u_2v_2) - 3u_1v_2(R_1+R_2)) + 4m^2u_1R + 2m^2u_2(R_1+R_2) \\
&- mu_1(R_2^2+4R_1R_2+R_1^2) - 4mu_2R(R_2+2R_1) + u_2(R_1^3+6R_1^2R_2+3R_1R_2^2)\}.
\end{aligned}
$$

while X_8^i $(i=3,\dots,6)$ can be derived from X_7^j $(j=3,\dots,6)$ by the transformation [18]

$$
\begin{aligned}
T : &u_1 \to u_2,\ u_2 \to u_1,\ v_1 \to v_2,\ v_2 \to v_1,\ \partial_x \to -\partial_x \\
&(R_1 \to R_2,\ R_2 \to R_1,\ R \to R)
\end{aligned}
\tag{4.3.15}
$$

in the following way

$$
X_8^3 = -T(X_7^4);\ X_8^4 = -T(X_7^3);\ X_8^5 = -T(X_7^6);\ X_8^6 = -T(X_7^5) \tag{4.3.16}
$$

The Lie bracket for the vertical vector fields X_i defined by

$$
X_i = X_i^3\partial_{u_1} + X_i^4\partial_{u_2} + X_i^5\partial_{v_1} + X_i^6\partial_{v_2} + \text{pr.} \tag{4.3.17}
$$

is given by

$$
[X_j,X_k]^l = L_{X_j}(X_k^l) - L_{X_k}(X_j^l) \quad (l = 3,\dots,6) \tag{4.3.18}
$$

In (4.3.18) only the $\partial_{u_1},\dots,\partial_{v_2}$ components of the commutator of two vector fields are defined, while the other components are derived by total differentiation.

Computation of (4.3.18) for the vector fields $X_1,\dots,X_8$ given in (4.3.7) (4.3.11), (4.3.14) and (4.3.16) results in the following nonzero commutators

$$[X_1,X_3] = X_1;\; [X_2,X_3] = -X_2;$$
$$[X_3,X_3] = -2X_5 - \frac{m^2}{2} X_4;\; [X_3,X_6] = 2X_6 - \frac{m^2}{2} X_4; \qquad (4.3.19)$$
$$[X_3,X_7] = -3X_7 + m^2(X_1+X_2);\; [X_3,X_8] = 3X_8 - m^2(X_1+X_2).$$

Transformation of the basis-vector fields of the Lie algebra by

$$Y_1 = X_1;\; Y_2 = X_2,\; Y_3 = X_3;\; Y_4 = X_4$$
$$Y_5 = X_5 + \frac{m^2}{4} X_4;\; Y_6 = X_6 - \frac{m^2}{4} X_4 \qquad (4.3.20)$$
$$Y_7 = X_7 - \frac{m^2}{2} X_1 - \frac{m^2}{4} X_2;\; Y_8 = X_8 - \frac{m^2}{4} X_1 - \frac{m^2}{2} X_2$$

then leads to the following table

j →

$[Y_i,Y_j]$	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8
Y_1	*	0	Y_1	0	0	0	0	0
Y_2		*	$-Y_2$	0	0	0	0	0
Y_3			*	0	$-2Y_5$	$2Y_6$	$-3Y_7$	$3Y_8$
Y_4				*	0	0	0	0
Y_5					*	0	0	0
Y_6						*	0	0
Y_7							*	0
Y_8								*

i ↓

table 4.3.1.

Note that from (4.3.19) and table 1 we see $[Y_i,Y_j] = 0$ $(i,j=1,2,5,6,7,8)$ i.e., the Lie Bäcklund transformations commute.

4.4 Nonlocal Lie-Bäcklund transformations of the Massive Thirring Model

In this section we give a short introduction to the notion of nonlocal Lie-Bäcklund transformations and construct nonlocal Lie-Bäcklund transformations of the Massive Thirring Model.

First, let us consider the KdV-equation

$$u_t = uu_x + u_{xxx}. \tag{4.4.1}$$

The KdV-equation admits an infinite hierarchy of commuting Lie-Bäcklund transformations obtained by the action of the Lenard recursion operator (1.2.20) on the Lie-Bäcklund transformations

$$\begin{aligned} V_1 &= u_x\partial_u + \ldots \\ V_2 &= (u_{xxx} + u_x u)\partial_u + \ldots \end{aligned} \tag{4.4.2}$$

The vector fields V_1, V_2 are equivalent to the infinitesimal symmetries [12]

$$\bar{V}_1 = -\partial_x \ , \ \bar{V}_2 = -\partial_t$$

The Lie algebra of infinitesimal symmetries of (4.4.1) is 4-dimensional and generated by [12]

$$\begin{aligned} &\bar{V}_1 = -\partial_x, \ \ \bar{V}_2 = -\partial_t, \ \bar{V}_3 = x\partial_x + 3t\partial_t - 2u\partial_u + \ldots \\ &\bar{V}_4 = t\partial_x + \partial_u + \ldots \end{aligned} \tag{4.4.3}$$

If we formally apply the Lenard recursion operator $\mathcal{D}$ on the generating function of the Lie-Bäcklund transformation

$$V_3 = \{-2u - xu_x - 3t(uu_x + u_{xxx})\}\ \partial_u + \ldots, \tag{4.4.4}$$

V_3 being equivalent to $\bar{V}_3$, we obtain the following result

$$\begin{aligned} \eta = &-x(u_{xxx}+u_x u) - 3t(u_{xxxxx} + \tfrac{5}{3}\, u_{xxx}u + \tfrac{10}{3}\, u_{xx}u_x + \tfrac{5}{6}\, u_x u^2) \\ &-4u_{xx} - \tfrac{4}{3}\, u^2 - \tfrac{1}{3}\, u_x D^{-1} u \end{aligned} \tag{4.4.5}$$

where D^{-1} is defined by (4.2.3)

This motivates the introduction of a nonlocal variable

$$p = D^{-1}u = \int_{-\infty}^{x} udx \tag{4.4.6}$$

Note that p is a potential of the KdV equation or

$$dp - udx - (u_{xx} + \frac{1}{2}u^2)dt \tag{4.4.7}$$

is a potential form.

In [20], Krasilshchik & Vinogradov gave a nice discussion of a theory of nonlocal symmetries and nonlocal Lie-Bäcklund transformations introducing coverings.

The Estabrook-Wahlquist prolongation method by means of pseudo potentials [40], fits within this theory. The construction of Krasilshchik & Vinogradov is based on the prolongation of total derivative vector fields (1.1.12), (1.2.5) towards nonlocal variables in such a way that the prolonged vector fields still commute within the more general setting.

In the application to the Massive Thirring Model we are only interested in prolongations arising from potential forms.

Since the notions of infinitesimal symmetries and Lie-Bäcklund transformations were introduced using exterior differential systems, we describe nonlocal Lie-Bäcklund transformations in a way similar to sections 2,3 of chapter 1, leading in effect to the same conditions as those obtained in [20].

We prolong the ideal of differential forms $D^{\infty}I$, describing a differential equation on the infinite jet bundle J(M,N) by potential forms associated to infinitesimal symmetries of I (1.1.24), (1.1.38) i.e., $P_0, P_2, \ldots, P_k$.

$$P_i = dp_i + \text{conserved current } (i=0,\ldots,k) \tag{4.4.8}$$

We now extend the definition of a Lie-Bäcklund transformation (1.2.16), (1.2.17) to nonlocal variables $p_1, p_2, \ldots$ by requiring that the components of the vector field V depending on $p_1, p_2, \ldots$ too and satisfy the condition (c.f. 1.2.17)

$$L_V I \subset \langle D^r I,\ P_0, \ldots, P_k \rangle \quad \text{for some } r. \tag{4.4.9}$$

Condition (4.4.9) leads to conditions on the local components of the vector field V (c.f. chapter 1, section 3).
Since

$$dP_i \in I \; , \qquad (i=0,\dots,k) \tag{4.4.10}$$

we do not need to take into account the exterior derivatives dP_i $(i=1,\dots,k)$ in the right hand side of (4.4.9).
The nonlocal components of the vector field V have to satisfy the condition

$$L_V P_i \in \langle D^r I, \; P_o,\dots,P_k \rangle \quad \text{for some } r. \tag{4.4.11}$$

It will turn out that the nonlocal Lie-Bäcklund transformations obtained in this section do not admit prolongation to the nonlocal components $\partial_{P_i},\dots$ in the sense of (4.4.11).
The way out of this problem will presumably be to take into account higher order potential forms associated to Lie-Bäcklund transformations.

We now construct nonlocal Lie-Bäcklund transformations of the Massive Thirring Model. (4.3.1)
First of all we introduce a Lagrangian L for the Massive Thirring Model i.e.,

$$\begin{aligned} &L(u_1,\dots,v_2,u_{1x},\dots,v_{2t}) = \\ &\tfrac{1}{2}\{- u_1 v_{1x} + u_{1x} v_1 + u_2 v_{2x} - u_{2x} v_2 \\ &\quad + u_1 v_{1t} - u_{1t} v_1 + u_2 v_{2t} - u_{2t} v_2\} + \\ &\quad + m(u_1 u_2 + v_1 v_2) - \tfrac{1}{2}(u_1^2+v_1^2)(u_2^2+v_2^2). \end{aligned} \tag{4.4.12}$$

A straight-forward computation shows that the Euler-Lagrange equations associated to (4.4.12) are just the system of partial differential equations (4.3.1)

Application of Noether's Theorem to the infinitesimal symmetries

$$\begin{aligned} &0_1 = \partial_x \qquad 0_2 = \partial_t \\ &0_3 = v_1 \partial_{u_1} + v_2 \partial_{u_2} - u_1 \partial_{v_1} - u_2 \partial_{v_2} \\ &0_4 = t\partial_x + x\partial_t - \tfrac{1}{2} u_1 \partial_{u_1} + \tfrac{1}{2} u_2 \partial_{u_2} - \tfrac{1}{2} v_1 \partial_{v_1} + \tfrac{1}{2} v_2 \partial_{v_2} \end{aligned} \tag{4.4.13}$$

which are equivalent to $X_1,\dots,X_4$ (4.3.7), leads to the following conserved vectors

$$\begin{aligned}
A_1^x &= \frac{1}{2}\{u_1v_{1x} - u_{1x}v_1 - u_2v_{2x} + u_{2x}v_2 + R_1R_2\} \\
A_1^t &= \frac{1}{2}\{-u_1v_{1x} + u_{1x}v_1 - u_2v_{2x} + u_{2x}v_2\} \\
A_2^x &= \frac{1}{2}\{u_1v_{1x} - u_{1x}v_1 + u_2v_{2x} - u_{2x}v_2\} \\
A_2^t &= \frac{1}{2}\{-u_1v_{1x} + u_{1x}v_1 + u_2v_{2x} - u_{2x}v_2 - R_1R_2 + 2mR\} \\
A_3^x &= \frac{1}{2}\{R_1-R_2\} \\
A_3^t &= -\frac{1}{2}\{R_1+R_2\} \qquad (4.4.14) \\
A_4^x &= \frac{1}{2}x\{u_1v_{1x} - u_{1x}v_1 + u_2v_{2x} - u_{2x}v_2\} \\
&\quad + \frac{1}{2}t\{u_1v_{1x} - u_{1x}v_1 - u_2v_{2x} + u_{2x}v_2 + R_1R_2\} \\
A_4^t &= \frac{1}{2}x\{-u_1v_{1x} + u_{1x}v_1 + u_2v_{2x} - u_{2x}v_2 - R_1R_2 + 2mR\} \\
&\quad + \frac{1}{2}t\{-u_1v_{1x} + u_{1x}v_1 - u_2v_{2x} + u_{2x}v_2\}
\end{aligned}$$

where R, R_1, R_2 are defined by (4.3.12)

Our first attempt, without success, in searching for a generating Lie-Bäcklund transformation was an (x,t)-dependent local Lie-Bäcklund transformation of degree 2, because X_1, X_2 (4.3.7) are of degree 2, X_5, X_6 (4.3.11) being of degree 4.
We were motivated by the form of the nonlocal Lie-Bäcklund transformations for the KdV-equation (4.4.1), (4.4.5) [7].
Moreover, the generating (local) Lie-Bäcklund transformations of Burgers' equation [39], (4.1.18)

$$\phi = (x(2u_xu + 2u_x) + t(4u_{xxx} + 6u_{xx}u + 6u_x^2 + 3u_xu^2) + u^2)\partial_u + \dots \qquad (4.4.15)$$

and of the Classical Boussinesq equation (4.2.14)
$(u_t=uv_x+u_xv+\sigma v_{xxx};v_t=u_x+v_xv)$,

$$Z_1 = \{t(\sigma u_{xxx} + \frac{3}{2}\sigma v_{xxx}v + 3\sigma v_{xx}v_x + \frac{3}{4}u_x(v^2+2u) + \frac{3}{2}v_xvu) + + \frac{1}{2}x(\sigma v_{xxx}+u_xv+uv_x) + \frac{3}{2}\sigma v_{xx}+uv\}\,\partial_u + \{t(\sigma v_{xxx}+\frac{3}{2}u_xv + \frac{3}{4}v_x(v^2+2u) + \frac{1}{2}x(u_x+v_xv) + \frac{v^2}{4} + u\}\partial_v + \ldots \quad (4.4.16)$$

which is obtained by the action of the generating operator $\mathcal{D}$ on the scaling are linear in x and t. Moreover the coefficients of x and t in (4.4.15), (4.4.16) are Lie-Bäcklund transformations themselves.

Motivated by these observation we introduce nonlocal variables p_0, p_1, p_2 by the potential forms

$$P_0 = dp_0 - p_0^1dx - p_0^2dt; \quad P_1 = dp_1 - p_1^1dx - p_1^2dt; \quad P_2 = dp_2 - p_2^1dx - p_2^2dt; \quad (4.4.17)$$

where

$$p_0^1 = -A_3^t, \quad p_1^1 = -(A_1^t + A_2^t), \quad p_2^1 = -(A_1^t - A_2^t), \quad p_0^2 = +A_3^x, \quad p_1^2 = A_1^x + A_2^x, \quad p_2^2 = A_1^x - A_2^x. \quad (4.4.18)$$

We now construct the ideal of differential forms I'

$$I' = \langle D^3I, P_0, P_1, P_2\rangle, \quad (4.4.19)$$

and impose the condition (4.4.9)

$$L_V I \subset \langle D^3I.P_0, P_1, P_2\rangle, \quad (4.4.20a)$$

which does lead to conditions on the local components of the vector field V; components which are supposed to depend on

$$p_0, p_1, p_2.x, t, u_1, \ldots v_2, \ldots, u_{1xx}, \ldots v_{2xx}. \quad (4.4.20b)$$

The resulting conditions on the local components of V are similar to the conditions obtained by prolongation of the total derivative operators D_x, D_t i.e., $\tilde{D}_x, \tilde{D}_t$.

Motivated by the results for KdV, Burgers' and Classical Boussinesq equation we search for a Lie-Bäcklund transformation

$$V = x\ LB_1 + t\ LB_2 + C \tag{4.4.21}$$

where LB_1 and LB_2 are (x,t)-independent Lie-Bäcklund transformations of degree $\leqslant 4$, while C has to be of degree 2.

(Note that the components have to be of degree 5 and 3 respectively). Since in this specific problem the mass m is of degree 2, we take LB_1, LB_2 to be linear combinations of $X_1, \ldots, X_6$, (4.3.7) (4.3.11) whereas the C-components (2.12) are supposed to be linear in $u_{1x}, u_{2x}, v_{1x}, v_{2x}$.

Substitution of (4.4.21) into the overdetermined system of partial differential equations obtained from (4.4.20) and solving the resulting system leads to two (x,t)-dependent nonlocal Lie-Bäcklund transformations i.e,

$$Z_1^3 = v_1 p_2 + x[-2X_5^3 - m^2 v_1] + t[2X_5^3] + \frac{1}{2}\ mu_2,$$

$$Z_1^4 = v_2 p_2 + x[-2X_5^4 - m^2 v_2] + t[2X_5^4] + \frac{3}{2}\ mu_1 + 3v_{2x} - \frac{3}{2}\ R_1 u_2 - \frac{1}{2}\ R_2 u_2,$$

$$Z_1^5 = -u_1 p_2 + x[-2X_5^5 + m^2 u_1] + t[2X_5^5] + \frac{1}{2}\ mv_2,$$

$$Z_1^6 = -u_1 p_2 + x[-2X_5^6 + m^2 u_2] + t[2X_5^6] + \frac{3}{2}\ mv_1 - 3u_{2x} - \frac{3}{2}\ R_1 v_2 - \frac{1}{2}\ R_2 v_2,$$

and

$$Z_2^3 = v_1 p_1 + x[-2X_6^3 + m^2 v_1] + t[-2X_6^3] + \frac{3}{2}\ mu_2 - 3v_{1x} - \frac{3}{2}\ R_2 u_1 - \frac{1}{2}\ R_1 u_1, \tag{4.4.22}$$

$$Z_2^4 = v_2 p_1 + x[-2X_6^4 + m^2 v_2] + t[-2X_6^4] + \frac{1}{2}\ mu_1,$$

$$Z_2^5 = -u_1 p_1 + x[-2X_6^5 - m^2 u_1] + t[-2X_6^5] + \frac{3}{2}\ mv_2 + 3u_{1x} - \frac{3}{2}\ R_2 v_1 - \frac{1}{2}\ R_1 v_1,$$

$$Z_2^6 = -u_2 p_1 + x[-2X_6^6 - m^2 u_2] + t[-2X_6^6] + \frac{1}{2}\ mv_1.$$

Note that the local components of the vector fields Z_1 and Z_2 do not depend on the nonlocal variable p_0.

From now on we discard p_0 from our considerations.

In orde to derive the action of the vector fields Z_1 and Z_2 on the vector fields $X_1,\dots,X_6$ (4.4.22), (4.3.7), (4.3.11) we have to extend the Lie bracket in a way analogous to Krashilchik & Vinogradov [20].

The nonlocal components of the vector fields $X_1,\dots,X_6$ are obtained by prolongation, expressed by the condition (4.4.11)

$$L_{X_j} P_i \in \langle D^3 I, P_1, P_2 \rangle, \quad \begin{matrix}(i=1,2)\\(j=1,\dots,6)\end{matrix} \tag{4.4.23}$$

This condition is equivalent to the condition that the Lie derivative L_{X_j} of the potential equations

$$\begin{aligned} p_{1x} + (A_1^t + A_2^t) &= 0 \qquad & p_{1t} - (A_1^x + A_2^x) &= 0 \\ p_{2x} + (A_1^t - A_2^t) &= 0 \qquad & p_{2t} - (A_1^1 - A_2^1) &= 0 \end{aligned} \tag{4.4.24}$$

is zero subject to (4.3.1), (4.4.24) and their differential consequences.

We shall not take into account integration constants arising from condition (4.4.23). They refer to symmetries ∂_{p_1}, ∂_{p_2} of (4.3.1), (4.4.24)

The computation of (4.4.23) leads to the following nonlocal components of the vector field $X_1,\dots,X_6$

$$X_j = X_j^1\partial_{p_1} + X_j^2\partial_{p_2} + X_j^3\partial_{u_1} + X_j^4\partial_{u_2} + X_j^5\partial_{v_1} + X_j^6\partial_{v_2} \tag{4.4.25a}$$

$$X_1^1 = -\frac{1}{2}\, m\, R$$

$$X_1^2 = -v_2u_{2x} + u_2v_{2x} + \frac{1}{2}\, mR - \frac{1}{2}\, R_1R_2$$

$$X_2^1 = -v_1u_{1x} + u_1v_{1x} - \frac{1}{2}\, mR + \frac{1}{2}\, R_1R_2$$

$$X_2^2 = \frac{1}{2}\, mR$$

$$X_3^1 = \frac{1}{2}\,(x+t)(+2u_1v_{1x} - 2v_1u_{1x} + R_1R_2) - mtR + p_1$$

$$X_3^2 = \frac{1}{2}\,(x+t)(-2u_2v_{2x} + 2v_2u_{2x} + R_1R_2) + mtR - p_2$$

$$X_4^1 = 0 \qquad X_4^2 = 0$$

$$X_5^1 = -\frac{1}{2} mv_1u_{2x} + \frac{1}{2} mu_1v_{2x} - \frac{1}{4} mR(R_1+R_2) + \frac{1}{4} m^2(R_1+R_2)$$

$$\begin{aligned} X_5 = &+ u_{2xx}u_2 + v_{2xx}v_2 - {}^2u_{2x} - {}^2v_{2x} \\ &- \frac{1}{2} mu_2v_{1x} + mv_1u_{2x} + \frac{1}{2} mv_2u_{1x} - mu_1v_{2x} \\ &- u_{2x}(R_2v_2 + 2R_1v_2) + v_{2x}(R_2u_2 + 2R_1u_2) \\ &- \frac{1}{4} m^2(R_1 + R_2) + \frac{3}{4} mR(R_1+R_2) + \frac{1}{2} R_1R_2(R_1+R_2) \end{aligned} \qquad (4.4.25b)$$

$$\begin{aligned} X_6^1 = &- u_{1xx}u_1 - v_{1xx}v_1 + v_{1x}^2 + u_{1x}^2 \\ &- \frac{1}{2} mv_{2x}u_1 + mu_{1x}v_2 + \frac{1}{2} mu_{2x}v_1 - mv_{1x}u_2 \\ &- u_{1x}v_1(R_1+2R_2) + v_{1x}u_1(R_1+2R_2) \\ &+ \frac{1}{4} m^2(R_1+R_2) - \frac{3}{4} mR(R_1+R_2) - \frac{1}{2} R_1R_2(R_1+R_2) \end{aligned}$$

$$X_6^2 = -\frac{1}{2} mv_2u_{1x} + \frac{1}{2} mu_2v_{1x} + \frac{1}{4} mR(R_1+R_2) - \frac{1}{4} m^2(R_1+R_2)$$

While the p_1 component of Z_1 and the p_2-component of Z_2 are given by

$$\begin{aligned} Z_1^1 &= \frac{1}{2} (x-t)\{-2mu_1v_{2x} + 2mv_1u_{2x} - (-m^2+mR)(R_1+R_2)\} - \frac{1}{2}mu_1v_2 + \frac{1}{2}mu_2v_1 \\ Z_2^2 &= \frac{1}{2} (x+t)\{-2mu_2v_{1x} + 2mv_2u_{1x} + (m^2-mR)(R_1+R_2)\} + \frac{1}{2}mu_1v_2 - \frac{1}{2}mu_2v_1 \end{aligned} \qquad (4.4.26)$$

Computation of the generalized Lie-bracket then leads to the following result

$$\begin{aligned} [Z_1,X_1] &= -\frac{1}{2} m^2X_4 - 2X_5 & [Z_2,X_1] &= \frac{1}{2} m^2X_4 \\ [Z_1,X_2] &= -\frac{1}{2} m^2X_4 & [Z_2,X_2] &= \frac{1}{2} m^2X_4 - 2X_6 \\ [Z_1,X_3] &= Z_1 & [Z_2,X_3] &= -Z_2 \\ [Z_1,X_4] &= 0 & [Z_2,X_4] &= 0 \\ [Z_1,X_5] &= 4X_7 - 2m^2X_1 - m^2X_2 & [Z_2,X_5] &= m^2X_1 \\ [Z_1,X_6] &= m^2X_2 & [Z_2,X_6] &= 4X_8 - m^2X_1 - 2m^2X_2; \end{aligned} \qquad (4.4.27)$$

while

$$[Z_1,Z_2] = -2m^2X_3.$$

Transformation of the basis vector fields by

$$Y_1 = X_1,\ Y_2 = X_2,\ Y_3 = X_3,\ Y_4 = X_4$$

$$Y_5 = X_5 + \frac{m^2}{4} X_4,\ Y_6 = X_6 - \frac{m^2}{4} X_4 \qquad (4.4.28)$$

$$Y_7 = Y_7 - \frac{m^2}{2} X_1 - \frac{m^2}{4} X_2,\ Y_8 = X_8 - \frac{m^2}{4} X_1 - \frac{m^2}{2} X_2$$

yields the following commutators

$$\begin{array}{ll} [Z_1,Y_1] = -2Y_5 & [Z_2,Y_1] = \frac{1}{2} m^2 Y_4 \\ [Z_1,Y_2] = -\frac{1}{2} m^2 Y_4 & [Z_2,Y_2] = -2Y_6 \\ [Z_1,Y_3] = Z_1 & [Z_2,Y_3] = -Z_2 \\ [Z_1,Y_4] = 0 & [Z_2,Y_4] = 0 \\ [Z_1,Y_5] = 4Y_7 & [Z_2,Y_5] = m^2 Y_1 \\ [Z_1,Y_6] = m^2 Y_2 & [Z_2,Y_6] = 4Y_8, \end{array} \qquad (4.4.29)$$

while

$$[Z_1,Z_2] = -2m^2 Y_3$$

From (4.4.29) we conclude that Z_1 acts as a generating operator on Y_1, Y_5, while Z_2 acts as a generating operator on Y_2, Y_6.
The action of Z_1 on Y_2, Y_6 is annihilating, just as Z_2 acts on Y_1, Y_5.

We suspect that the vector fields Z_1 and Z_2 generate a hierarchy of commuting Lie-Bäcklund transformations.

Remark

In (4.4.29) only Z_1^1, Z_2^2 are given, necessary to compute the generalized Lie bracket

$$[Z_1,Z_2] = -2m^2 Y_3$$

We should mention that Z_1 does not admit a prolongation Z_1^2 while Z_2 does not admit a Z_2^1 prolongation in this formulation (4.4.23).

Probably they do admit a prolongation in a more general formulation, taking into account higher order nonlocal variables related to the Lie-Bäcklund transformations Y_5, Y_6 (4.3.11)

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SOURCE CODE

```
PROCEDURE LIEDF(FEC,FORM);
IP(FEC,EXDF( FORM))+EXDF(IP(FEC,FORM))$

LISP OPERATOR LENGTH,CLEARKVALUE$
LISP PROCEDURE CLEARKVALUE(OP);
<<REMPROP(OP,'KVALUE);0>>$

OPERATOR ALFA,CO,VER;
LISP FLAG('(CO),'NOHASH)$
PROCEDURE INFSYM(PKVAN,PKTOT,N1);
BEGIN
%C1
   SCALAR VEC,LD,AA,AANT,LL;
   INTEGER MM,MM1,JJ,II,KK;
   ARRAY RA N1;
%C2
   FOR II:=1:N1 DO ALFA II:=NORMDIF(ALFA II);
   MM:=1;MM1:=0;
%C3
   VEC:=(FOR II:=1:D!@DIF  SUM F(II)*D^II);
%C4
   FOR JJ:=1:N1 DO
   BEGIN
      AA:=!@BEVATOP(ALFA(JJ),UIT);
      RA(JJ):=LENGTH AA-1;
   END;
%C5
   FOR II:=PKVAN:PKTOT DO
   BEGIN
%C5.1
      LD:=LIEDF(VEC,ALFA II)-(FOR JJ:=1:N1 SUM
          MULFORM(GEFORM(JJ,RA II-RA JJ),ALFA JJ));
%C5.2
      AANT:=OPCOEFF(LD,UIT,C);
%C5.3
      FOR KK:=0:AANT DO
      BEGIN
         L1PKINF:=OPCOEFF(C(1,KK),CO,OPKINF);
         SPKINF:=0;
         WHILE SPKINF < L1PKINF+1 DO
         IF (OPKINF(1,SPKINF)^2)=1
         THEN
            BEGIN
               OPL(NUM C(1,KK),OPKINF(0,SPKINF));
               SPKINF:=L1PKINF+1;
            END
         ELSE SPKINF:=SPKINF+1;
         END;
%C5.4
      FOR KK:=0:AANT DO
      BEGIN
         LL:=!@BEVATOP(C(1,KK),CO);
         IF LL NEQ 0
```

```
          THEN
             BEGIN
                WRITE "CREATION OF DENOMINATOR !!";
                OPL(NUM C(1,K),LL)
             END
          ELSE IF C(1,KK) NEQ 0
                 THEN
                    BEGIN VER MM:=C(1,KK);
                          VER MM:=VER MM*DEN VER MM;
                          MM:=MM+1;
                    END;
       END;
       TOTAL:=MM-1;
       WRITE "THERE EXIST ",MM-1-MM1," EQUATIONS FOR ALFA(",II,")";
       MM1:=MM-1;CLEARKVALUE(CO);
    END;
    RETURN MM-1;
END$

PROCEDURE GEFORM(NR,J);
IF J=0
THEN  CO(NR,J)
ELSE
   IF J=1
   THEN  (FOR I:=1:D!@DIF SUM CO(NR,J,I)*UIT I)
   ELSE
      IF J=2
      THEN   (FOR I:=1:D!@DIF SUM
             (FOR K:=I+1:D!@DIF SUM CO(NR,J,I,K)*UIT(I,K)))
      ELSE
         IF J=3
         THEN  (FOR I:=1:D!@DIF SUM
               (FOR K:=I+1:D!@DIF SUM
               (FOR L:=K+1:D!@DIF SUM CO(NR,J,I,K,L)*UIT(I,K,L))))
         ELSE 0$

ARRAY FIPKA 20,FIPKA1 0;
WRITE "ARRAY FIPKA HAS DIMENSION : 20"$

LISP OPERATOR FSOLV$
LISP PROCEDURE FSOLV(NUPKEQ,NUPKF);
BEGIN
%C1
      OPL(REVAL LIST('NUM,LIST('VER,NUPKEQ)),LIST('F,NUPKF));
%C2
      SETK(LIST('VER,NUPKEQ),0);
%C3
      DEPL!*:=DELETE(ASSOC(LIST('F,NUPKF),DEPL!*),DEPL!*);
END$

LISP GLOBAL '(VFPKTEL VFPKTRL FPKTEL)$
SHARE VFPKTEL,VFPKTRL,FPKTEL;
VFPKTEL:=0$FPKTEL:=0$

OPERATOR D,VF;
```

```
LISP VFPKTRL:=NIL$
LISP OPERATOR VFGEN$
OPERATOR PKVFV;
LISP PROCEDURE VFGEN(DIMVF);
BEGIN
%C1
SCALAR FLY,L1,L2,L3,L4,L5,L6,L7,L8,L10,L11,L12,L13,L14,L15,GAVER;
       L1:=NIL;L2:=NIL;L3:=NIL;L4:=NIL;L5:=NIL;
       L6:=NIL;L7:=NIL;L8:=NIL;GAVER:=T;
       L10:=NIL;L11:=NIL;L12:=NIL;L13:=NIL;L14:=NIL;L15:=NIL;
   FOR II:=1:DIMVF DO SETK(LIST('PKVFV,II),AEVAL LIST('F,II));
   FLY:=NIL;FOR EACH EL IN DEPL!* DO
   IF CAAR(EL)='F THEN <<FLY:=CAR(EL).FLY;WRITE CAR(EL);TERPRI()>>;
%C2
   IF VFPKTRL AND LENGTH(VFPKTRL)=DIMVF
   THEN L1:=VFPKTRL
   ELSE
      <<FOR II:=1:DIMVF DO L2:=II.L2;
        FOR EACH EL IN L2 DO L1:=LIST('X,EL).L1;
        GAVER:=NIL
      >>;

   FOR EACH EL IN FLY DO
   BEGIN
      L3:=DELETE(EL,FLY);
      FOR EACH EL1 IN L3 DO SETK(EL1,0);
%C3.1
      IF LENGTH (ASSOC(EL,DEPL!*))=1
      THEN SETK(EL,1);
%C3.2
      FOR I:=1:DIMVF DO
      L4:=LIST('TIMES,REVAL LIST('PKVFV,I),LIST('D,NTH(L1,I))).L4;
      IF (L15:=AEVAL('PLUS.L4))=0
      THEN <<WRITE EL," IS NOT PRESENT";TERPRI()>>
      ELSE SETK(LIST('VF,(VFPKTEL:=VFPKTEL+1)),L15);
      CLEARKVALUE 'F;L4:=NIL
   END;
   IF GAVER
   THEN <<L7:=L1;FOR I:=1:DIMVF DO
              <<SETK(LIST('X,I),CAR L7);L7:=CDR L7>>;
%C4
         L10:=NIL;
         FOR EACH EL IN DEPL!* DO
         IF LENGTH (EL)=1
         THEN L10:=EL.L10
         ELSE <<L11:=CDR EL;L12:=CAR EL;
                L13:=NIL;
                FOR EACH EL1 IN L11 DO L13:=(REVAL EL1).L13;
                L14:=NIL;
                FOR EACH EL2 IN L13 DO L14:=EL2.L14;
                L14:=L12.L14;L10:=L14.L10;
              >>;
         DEPL!*:=L10
       >>;
    FOR EACH EL IN L1 DO L5:=EL.L5;
    FOR EACH EL IN L5 DO L6:=LIST('D,EL).L6;
```

```
%C5
   FACTORS!*:=MAPCAR(L6,FUNCTION !*A2K);
   ORDL!*:=MAPCAR(L6,FUNCTION !*A2K);
   WRITE
   "CREATION OF A TOTAL NUMBER OF ",VFPKTEL,"  VECTOR FIELDS";
   TERPRI();
END$

LISP OPERATOR ELIM$
LISP PROCEDURE ELIM(ELIM1,ELIM2);
BEGIN
%C1
   SCALAR L1,L2,L3,L4,L6,S1,GAVER;
   GAVER:=T;
   L1:=ASSOC(LIST('F,ELIM1),DEPL!*);
   IF   (L2:=ASSOC(LIST('F,ELIM2),DEPL!*))
   THEN S1:=CDR L2;
%C2
   WHILE S1 DO
   IF CAR S1 MEMBER L1
   THEN S1:=CDR S1
   ELSE
      BEGIN
         S1:=NIL;
         GAVER:=NIL;
         WRITE "WRONG ELIMINATION";
         TERPRI();
      END;
%C3
   IF GAVER
   THEN
%C4  ELIMINATION PROCESS
      BEGIN
%C4.1
         L3:=CDR L1;
         L4:=LIST('F,(FPKTEL:=FPKTEL+1));
         L6:=LIST('DIFFERENCE,L4,LIST('F,ELIM2));
         DEPL!*:=DELETE(L1,DEPL!*);
         DEPL!*:=(L4.CDR L1).DEPL!*;
%C4.2
         SETK(LIST('F,ELIM1),AEVAL L6);
      END;
END$

WRITE "EXECUTE:ARRAY FIPKA1(MAX DIM);"$

LISP GLOBAL '(TOTAL FIDEPT);
SHARE TOTAL,FIDEPT;

FIDEPT:=0$
LISP OPERATOR FINES$
LISP PROCEDURE FINES (VERNUM);
BEGIN
%C0.1
SCALAR GAVERDER,GAVERDER1,TELVER,AANTDFVER,AANTFVER,
  MS4,NIETAANWARG,WELAANWARG,TOEGEDIFVAR,
```

```
  DS0,DS1,DS2,DS3,DS4,DS5,DS6,DS7,DS8,
  OS1,OS3,OS5,
  IS2,IS3,IS4,IS5,IS6,IS7,IS8,IS9,IS10,IS11,IS12,IS13,IS14,
  DD1,DD2,DD3,DD4,DD5,DD6,DD7,DD8,DD9,RESMET;
  IS5:=NIL;IS7:=NIL;;FIDEPT:=FIDEPT+1;
  GAVERDER:=T;GAVERDER1:=T;
%C0.2
IF VERNUM<=0 OR VERNUM>=(TOTAL+1)
THEN <<WRITE VERNUM," OUT OF RANGE :1,...,",TOTAL;
             TERPRI();GAVERDER:=NIL>>;
IF GAVERDER
%C0.3
THEN IF (TELVER:=REVAL LIST('NUM,LIST('VER,VERNUM)))=0
     THEN <<WRITE "VER(",VERNUM,") IS ZERO";TERPRI();GAVERDER:=NIL>>;
IF GAVERDER
THEN
%C0.4
<<AANTDFVER:=OPCOEFF(TELVER,'DF,'PKDFC);
%C0.5
  IF AANTDFVER=0 AND !@RESTOPCOEFF=0
  THEN
  <<
%C1   HOMOGENEOUS INTEGRATION(CASE B OF SECTION 2)
%C1.1
    DS0:=PKDFC(0,0);
    DS1:=CADR DS0;
    DS2:=ASSOC(DS1,DEPL!*);
    DS3:=CDR DS2;
    DS4:=CDDR DS0;
    DS8:=NIL;
    WRITE "HOMOGENEOUS INTEGRATION OF :",PKDFC(0,0);TERPRI();
%C1.2     CONSTRUCTION OF  TERMS
    WHILE DS4 DO
%C1.2.1
    <<DS5:=CAR DS4;
      DS7:=DELETE(DS5,DS3);
      DS4:=CDR DS4;
      IF DS4
      THEN
%C1.2.2
         IF NUMBERP(DS6:=CAR DS4)
         THEN
         <<DS4:=CDR DS4;
           FOR I:=2:DS6 DO
           <<DEPL!*:=(LIST('F,FPKTEL+1).DS7).DEPL!*;
             DS8:=LIST('TIMES,LIST('F,(FPKTEL:=FPKTEL+1)),
                  LIST('EXPT,DS5,I-1)).DS8;
           >>
         >>;
%C1.2.3
         DS8:=LIST('F,(FPKTEL:=FPKTEL+1)).DS8;
         DEPL!*:=(CAR(DS8).DS7).DEPL!*;
    >>;
%C1.3
     SETK(DS1,AEVAL('PLUS.DS8));
     SETK(LIST('VER,VERNUM),0);
```

```
     DEPL!*:=DELETE(ASSOC(DS1,DEPL!*),DEPL!*);
     GAVERDER:=NIL
   >>;
   IF GAVERDER
   THEN
   <<
%C0.6
     AANTFVER:=OPCOEFF(!@RESTOPCOEFF,'F,'PKFC);
%C0.7
     NIETAANWARG:=WELAANWARG:=CDR ASSOC('LIOV,DEPL!*);
     FOR II:=0:AANTDFVER
     DO BEGIN FOR EACH EL IN CDR ASSOC(CADR PKDFC(0,II),DEPL!*)
              DO NIETAANWARG:=DELETE(EL,NIETAANWARG);
        END;
     FOR II:=0:AANTFVER
     DO BEGIN FOR EACH EL IN CDR ASSOC(PKFC(0,II),DEPL!*)
              DO NIETAANWARG:=DELETE(EL,NIETAANWARG);
        END;
%C0.8
     IF NIETAANWARG THEN  <<STRPOLY(TELVER,NIETAANWARG);>>
     ELSE TEL1:=1;
%C0.9
     IF TEL1>1
     THEN
%C0.10
     <<SETK(LIST('VER,VERNUM),0);
       WRITE "VER(",VERNUM,") BREAKS INTO";
       WRITE " VER(",TOTAL+1,"),...,VER(",TOTAL+TEL1,") BY :";
       TERPRI();
       WRITE NIETAANWARG;TERPRI();
       FOR II:=(TOTAL+1):(TOTAL+TEL1)
       DO SETK(LIST('VER,II),REVAL LIST('HUVER,II-TOTAL));
       TOTAL:=TOTAL+TEL1;
       CLEARKVALUE('HUVER);
       TEL1:=0;
       TERPRI();
       GAVERDER:=NIL
     >>;
   >>;
>>;

IF GAVERDER
THEN
<<
%C2   SEARCH FOR A FUNCTION F(*)
%C2.1
  FOR EACH EL IN NIETAANWARG DO WELAANWARG:=DELETE(EL,WELAANWARG);
  TOEGEDIFVAR:=WELAANWARG;
  OS1:=NIL;
  FOR II:=0:AANTDFVER DO
%C2.2
  <<OS3:=ASSOC(CADR PKDFC(0,II),DEPL!*);
    IF LENGTH(CDR OS3)=LENGTH WELAANWARG
    THEN OS1:=CAR(OS3).OS1;
  >>;
  FOR II:=0:AANTFVER DO
```

```
%C2.3
  IF (OS5:=CDR ASSOC(PKFC(0,II),DEPL!*))=WELAANWARG
%C2.4
  THEN
     BEGIN
     IF NOT MEMBER(PKFC(0,II),OS1)
     THEN
%C2.5
       BEGIN
       WRITE "VER(",VERNUM,") IS SOLVED FOR : ",PKFC(0,II);
       TERPRI();
       FSOLV(VERNUM,CADR(PKFC(0,II)));
       TEL1:=0;II:=AANTFVER+1;GAVERDER:=NIL
       END
     END
%C2.6
  ELSE
     BEGIN
     FOR EACH EL IN OS5 DO TOEGEDIFVAR:=DELETE(EL,TOEGEDIFVAR);
     END;
>>;

%C3   SEARCH FOR A INHOMOGENEOUS INTEGRATION

IF GAVERDER
THEN
BEGIN
%C3.1
  IS2:=0;
  FOR II:=0:AANTDFVER DO  BEGIN
%C3.2
    IS3:=CADR PKDFC(0,II);
    IS4:=CDR ASSOC(IS3,DEPL!*);
%C3.3
    IF IS4=WELAANWARG
    THEN
    BEGIN
%C3.4
      IF IS2=0
      THEN <<IS2:=1;IS5:=II>>
      ELSE
        BEGIN
        WRITE "MORE THAN ONE MAXIMAL DF(F(*),*)";
        TERPRI();
        II:=AANTDFVER+1;
        GAVERDER:=NIL
        END;
    END
%C3.5
    ELSE<<FOR EACH EL IN IS4
        DO TOEGEDIFVAR:=DELETE(EL,TOEGEDIFVAR)>>
  END;
IF GAVERDER
%C3.6
THEN
  IF IS5
```

```
  THEN
 %C3.7
   <<IS6:=IS7:=CDDR PKDFC(0,IS5);
     WHILE IS6 DO
     IF MEMBER(CAR IS6,TOEGEDIFVAR)
     THEN <<IS6:=CDR IS6;
            IF IS6 THEN
            IF NUMBERP(CAR IS6)
            THEN IS6:=CDR IS6>>
     ELSE <<IS6:=NIL;GAVERDER1:=NIL>>;
     IF GAVERDER1
     THEN
 %C3.8
       IF NUMBERP(PKDFC(1,IS5))
       THEN
   <<
 %C3.9   THE CONSTRUCTION OF THE INHOMOGENEOUS PART
     WRITE "INHOMOGENEOUS INTEGRATION OF :",PKDFC(0,IS5);TERPRI();
 %C3.9.1
     IS8:=REVAL LIST('QUOTIENT,
     LIST('DIFFERENCE,LIST('TIMES,PKDFC(1,IS5),PKDFC(0,IS5)),TELVER),
     PKDFC(1,IS5));
 %C3.9.2
   WHILE  IS7
   DO
     <<IS9:=CAR IS7;
     IS7:=CDR IS7;
     IF IS7 AND  NUMBERP(CAR IS7)
     THEN <<IS10:=CAR IS7;IS7:=CDR IS7>>
     ELSE IS10:=1;
     IS14:=REVAL(LIST('DEN,IS8));
     IS11:=COEFF(REVAL LIST('NUM,IS8),IS9,'FIPKA);
     IS13:=NIL;
     FOR I:=0:IS11 DO
     <<IS12:=1;
       FOR J:=1:IS10 DO IS12:=(I+J)*IS12;
       IS13:=LIST('QUOTIENT,LIST('TIMES,REVAL GETEL LIST('FIPKA,I),
             LIST('EXPT,IS9,I+IS10)),IS12).IS13;
     >>;
     IS8:=REVAL(LIST('QUOTIENT,('PLUS.IS13),IS14));
   >>;
     IS8:=IS8.NIL;
 %C3.10   THE CONSTRUCTION OF THE HOMOGENEOUS PART
 %C1.1
     DS0:=PKDFC(0,IS5);
     DS1:=CADR DS0;
     DS2:=ASSOC(DS1,DEPL!*);
     DS3:=CDR DS2;
     DS4:=CDDR DS0;
 %C1.2  CONSTRUCTION OF  TERMS
     WHILE DS4 DO
 %C1.2.1
     <<DS5:=CAR DS4;
       DS7:=DELETE(DS5,DS3);
       DS4:=CDR DS4;
       IF DS4
```

```
      THEN
%C1.2.2
        IF NUMBERP(DS6:=CAR DS4)
        THEN
          << DS4:=CDR DS4;
             FOR I:=2:DS6 DO
            <<DEPL!*:=(LIST('F,FPKTEL+1).DS7).DEPL!*;
              IS8:=LIST('TIMES,LIST('F,(FPKTEL:=FPKTEL+1)),
              LIST('EXPT,DS5,I-1)).IS8;
            >>
          >>;
%C1.2.3
        IS8:=LIST('F,(FPKTEL:=FPKTEL+1)).IS8;
        DEPL!*:=(LIST('F,FPKTEL).DS7).DEPL!*;
    >>;
%C1.3
    SETK(DS1,AEVAL('PLUS.IS8));
%    SETK(LIST('VER,VERNUM),0);
    DEPL!*:=DELETE(ASSOC(DS1,DEPL!*),DEPL!*);
    GAVERDER:=NIL
 >>

%C3.11
  ELSE
     <<WRITE "THE COEFFICIENT OF THE DERIVATIVE IS NOT A NUMBER";
       TERPRI();GAVERDER:=NIL>>;
>>
END;
IF GAVERDER
THEN
IF FIDEPT=1
THEN
%C4   SEARCH FOR A DIFFERENTIATION (CASE E OF SECTION 2)
  <<WRITE "SEARCH FOR A DIFFERENTIATION";TERPRI();
%C4.1
    DD1:=NIL;DD2:=NIL;DD3:=CDR ASSOC('LIOV,DEPL!*);DD4:=1;
    FOR EACH EL IN DD3 DO
    <<DD2:=LIST(EL,DD4).DD2;DD4:=DD4+1;>>;
    FOR EACH EL IN DD2 DO DD1:=EL.DD1;
%C4.2
    FOR II:=0:AANTDFVER DO
    BEGIN
    FOR EACH EL1 IN CDR ASSOC(CADR PKDFC(0,II),DEPL!*) DO
       BEGIN
          DD9:=CADR ASSOC(EL1,DD1);
%C4.3
          SETEL(LIST('FIPKA1,DD9),
          GETEL(LIST('FIPKA1,DD9))+1);
       END;
    END;
%C4.4
    FOR II:=0:AANTFVER DO
    BEGIN
    DD6:=CDR ASSOC(PKFC(0,II),DEPL!*);
    FOR EACH EL1 IN DD6 DO
       BEGIN
```

```
          DD9:=CADR ASSOC(EL1,DD1);
          SETEL(LIST('FIPKA1,DD9),
          GETEL(LIST('FIPKAI,DD9))+1);
        END;
    END;
%C4.5
    RESMET:=FPKTEL;
    DD5:=LENGTH(ASSOC('LIOV,DEPL!*))-1;
    FOR II:=1:DD5 DO
    BEGIN
%C4.6
       IF GETEL(LIST('FIPKA1,II))=1
       THEN
%C4.7
       BEGIN
          DD7:=CAR NTH(DD1,II);WRITE DD7;TERPRI();
%C4.8
          DD8:=DEG(TELVER,DD7);
          SETK(LIST('VER,TOTAL+1),REVAL LIST('DF,TELVER,DD7,DD8+1));
          TOTAL:=TOTAL+1;
          WRITE "TOTAL:=",TOTAL;TERPRI();
%C4.9
          FINES TOTAL;
       END;
%C4.10
       SETEL(LIST('FIPKA1,II),0);
    END;
%C4.11
    IF (FPKTEL-RESMET)>0
    THEN
       BEGIN
          WRITE "ELIMINATION OF VARIABLE(S): FINES";
          TERPRI();
          TEL1:=0;
          FINES VERNUM;
       END;
>>;
TEL1:=0;
FIDEPT:=FIDEPT-1;
END$
CLEAR FIPKA1;

LISP FLAG('(F HUVER), 'NOHASH);
```

Subject index

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